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IDEALS OF HOLOMORPHIC FUNCTIONS WITH C^∞ BOUNDARY VALUES ON A PSEUDOCONVEX DOMAIN

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ABSTRACT. We give natural sufficient conditions for the solution of several problems concerning division in the space $\mathscr{A}^\infty(\Omega)$ of holomorphic functions with $\mathscr{C}^\infty$ boundary values on a pseudoconvex domain Ω with smooth boundary. The sufficient conditions come from upper semicontinuity with respect to the analytic Zariski topology of a local invariant of coherent analytic sheaves (the "invariant diagram of initial exponents"), and apply to division in the space of $\mathscr{C}^\infty$ Whitney functions on an arbitrary closed set. Our theorem on division in $\mathscr{A}^\infty(\Omega)$ follows using Kohn's theorem on global regularity in the $\bar\partial$-Neumann problem.

1. Introduction. Let Ω be a pseudoconvex domain with $\mathscr{C}^\infty$ boundary in $\mathbf{C}^n$, and let $\mathscr{A}^\infty(\Omega)$ denote the space of holomorphic functions on Ω with $\mathscr{C}^\infty$ boundary values. In this article, we give natural sufficient conditions for the solution of several problems concerning division in $\mathscr{A}^\infty(\Omega)$. Our sufficient conditions come from upper semicontinuity with respect to the analytic Zariski topology of a local invariant $\mathfrak{G}_a$ of coherent analytic sheaves (the "invariant diagram of initial exponents"). Semicontinuity of a coordinate-based diagram $\mathfrak{N}_a$ was first studied in [3], and used together with Hironaka's formal division algorithm [5] to give explicit solutions for problems concerning composition and division of $\mathscr{C}^\infty$ functions; Malgrange's division theorem [17, Chapter VI] is a classical special case. Semicontinuity of $\mathfrak{G}_a$ provides invariant sufficient conditions for division in the space of $\mathscr{C}^\infty$ Whitney functions on an arbitrary closed set. Our main theorem on division in $\mathscr{A}^\infty(\Omega)$ follows using Kohn's theorem on global regularity in the $\bar\partial$-Neumann problem [14].

We are indebted to Eric Amar for first bringing to our attention the questions considered here.

Let $\mathscr{C}^\infty(\bar\Omega)$ denote the Fréchet algebra of complex-valued $\mathscr{C}^\infty$ functions on $\bar\Omega$. Then $\mathscr{A}^\infty(\Omega)$ is a closed subalgebra of $\mathscr{C}^\infty(\bar\Omega)$. Let $\mathscr{O}(\bar\Omega)$ denote the ring of (germs at $\bar\Omega$ of) holomorphic functions defined in neighborhoods of $\bar\Omega$. Let A be a $p \times q$ matrix with entries in $\mathscr{O}(\bar\Omega)$. Then multiplication by A defines a continuous $\mathscr{A}^\infty(\Omega)$-homomorphism $A\colon \mathscr{A}^\infty(\Omega)^q \to \mathscr{A}^\infty(\Omega)^p$.

Under what conditions is $A \cdot \mathscr{A}^\infty(\Omega)^q$ a closed submodule of $\mathscr{A}^\infty(\Omega)^p$?

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Let $f^1, \ldots, f^q$ denote the columns of A; there is an open neighborhood U of $\bar{\Omega}$ such that each f^i is holomorphic in U. Let $\mathcal{O} = \mathcal{O}_U$ denote the sheaf of germs of holomorphic functions on U, and let $\mathcal{F}$ denote the sheaf of submodules of $\mathcal{O}^p$ generated by $f^1, \ldots, f^q$. The *invariant diagram of initial exponents* $\mathfrak{N}_a = \mathfrak{N}(\hat{\mathcal{F}}_a)$ is defined in §5 below. There is a natural total ordering on the set of all possible diagrams (see §2), so that semicontinuity as a function of $a \in U$ makes sense. §§4 and 5 establish a general theorem on the variation of the invariant diagram for certain parametrized families of modules of formal power series. The following is a special case (cf. Proposition 5.2):

THEOREM 1.1. *The invariant diagram $\mathfrak{N}_a$ is Zariski semicontinuous on U; i.e., there is a locally finite filtration of U by closed analytic subsets, $U = \Sigma_0(\mathcal{F}) \supset \Sigma_1(\mathcal{F}) \supset \cdots$, such that, for each $k = 0, 1, \ldots$,*

(1) *$\mathfrak{N}_a$ is constant, say $\mathfrak{N}_a = \mathfrak{N}_k(\mathcal{F})$, on $\Sigma_k(\mathcal{F}) - \Sigma_{k+1}(\mathcal{F})$;*

(2) *$\mathfrak{N}_k(\mathcal{F}) < \mathfrak{N}_{k+1}(\mathcal{F})$.*

For example, let X be a closed analytic subset of U and let $\mathcal{I}_X \subset \mathcal{O}$ denote the sheaf of germs of holomorphic functions which vanish on X. Then $\Sigma_1(\mathcal{I}_X) = X$ and $\Sigma_2(\mathcal{I}_X)$ is the complement in X of the smooth points of the highest dimension.

Let $a \in U$. Let $\hat{\mathcal{O}}_a$ denote the completion of $\mathcal{O}_a$ in the Krull topology. Then $\mathcal{O}_a$ (respectively, $\hat{\mathcal{O}}_a$) identifies with the ring of convergent (respectively, formal) power series $\mathbf{C}\{z\}$ (respectively, $\mathbf{C}[[z]]$), where $z = (z_1, \ldots, z_n)$. If $a \in \bar{\Omega}$, then there is a Taylor series homomorphism $f \mapsto \hat{f}_a$ from $\mathcal{A}^\infty(\Omega)^p$ to $\hat{\mathcal{O}}_a^p$. Let $(A \cdot \mathcal{A}^\infty(\Omega)^q)^\wedge$ denote the elements of $\mathcal{A}^\infty(\Omega)^p$ which *formally belong to* $A \cdot \mathcal{A}^\infty(\Omega)^q$; i.e.,

$$\left(A \cdot \mathcal{A}^\infty(\Omega)^q\right)^\wedge = \left\{f \in \mathcal{A}^\infty(\Omega)^p \colon \hat{f}_a \in \hat{A}_a \cdot \hat{\mathcal{O}}_a^q, \text{ for all } a \in \bar{\Omega}\right\}.$$

Clearly, $(A \cdot \mathcal{A}^\infty(\Omega)^q)^\wedge$ is a closed submodule of $\mathcal{A}^\infty(\Omega)^p$ which contains $A \cdot \mathcal{A}^\infty(\Omega)^q$.

THEOREM 1.2. *Suppose that $\Sigma_k(\mathcal{F})$ and $\bar{\Omega}$ are regularly situated, for all $k = 0, 1, \ldots$. Then*

$$A \cdot \mathcal{A}^\infty(\Omega)^q = \left(A \cdot \mathcal{A}^\infty(\Omega)^q\right)^\wedge;$$

in particular, $A \cdot \mathcal{A}^\infty(\Omega)^q$ is a closed submodule of $\mathcal{A}^\infty(\Omega)^p$.

We recall that two closed subsets X and Y of U are *regularly situated* if every point of U admits a neighborhood V and constants $c, r > 0$ such that $d(x, X) + d(x, Y) \geqslant c d(x, X \cap Y)^r$, for all $x \in V$ (Łojasiewicz; cf. [**22**, IV.4]). Any two closed real analytic (or even subanalytic) sets are regularly situated [**16**]. In particular, the conclusion of Theorem 1.2 always holds when $\partial\Omega$ is real analytic.

Previous articles, of De Bartolomeis-Tomassini [7], Amar [**1**] and Gay-Sebbar [**10**] have established Theorem 1.2 in the special case that $p = 1$ and the gradients of $f^1, \ldots, f^q$ are linearly independent in a neighborhood of $\partial\Omega$ (so that, in particular, $\Sigma_2(\mathcal{F}) \cap \partial\Omega = \varnothing$).[1]

[1] In a recent preprint, *Division, avec singularités, dans* $\mathcal{A}^\infty(\Omega)$ (Université de Bordeaux I, 1986), Amar gives a different version of our theorem in the case $p = 1$. For $p = q = 1$, he gives a filtration which is simpler than that of his general method, but coincides with that provided by semicontinuity of the diagram $\mathfrak{N}_a$ relative to certain weighted coordinates.

It can be important to find a continuous linear operator which solves the division problem of Theorem 1.2. Consider $A \cdot \mathscr{A}^\infty(\Omega)^q$ with the topology induced from $\mathscr{A}^\infty(\Omega)^p$. Under what conditions does the surjection $\mathscr{A}^\infty(\Omega)^q \to A \cdot \mathscr{A}^\infty(\Omega)^q$ admit a continuous linear splitting; i.e., a continuous linear operator $\lambda\colon A \cdot \mathscr{A}^\infty(\Omega)^q \to \mathscr{A}^\infty(\Omega)^q$ such that $f = A \cdot \lambda(f)$, for all $f \in A \cdot \mathscr{A}^\infty(\Omega)^q$?

An obvious necessary condition is that $A \cdot \mathscr{A}^\infty(\Omega)^q$ be a closed subspace of $\mathscr{A}^\infty(\Omega)^p$. We consider either of the following two conditions on Ω:

(a) $\partial\Omega$ is "weakly regular" in the sense of Catlin [**6**];

(b) $\partial\Omega$ is connected and Ω satisfies the conditions $Z(1)$, $Z(n-2)$ and $Z(n-1)$ of Folland and Kohn [**8**, p. 57].

Every bounded strictly pseudoconvex domain with $\mathscr{C}^\infty$ boundary is weakly regular.

COROLLARY 1.3. *Suppose that Ω is bounded and satisfies either condition* (a) *or* (b) *above. If each $\Sigma_k(\mathscr{F})$ is regularly situated with respect to $\overline{\Omega}$, then $A \cdot \mathscr{A}^\infty(\Omega)^q$ is a closed subspace of $\mathscr{A}^\infty(\Omega)^p$, and the canonical surjection $\mathscr{A}^\infty(\Omega)^q \to A \cdot \mathscr{A}^\infty(\Omega)^q$ admits a continuous linear splitting.*

Corollary 1.3 will be deduced from Theorem 1.2 in §9, using the theorem of Vogt and Wagner [**24**, **25**] on the splitting of exact sequences of Fréchet spaces. The division theorem 1.2 also provides a characterization of $\mathscr{A}^\infty$ functions which vanish on a closed analytic set:

Let X be a closed analytic subset of some neighborhood of $\overline{\Omega}$. Set $\mathscr{A}^\infty(\Omega; X) = \{f \in \mathscr{A}^\infty(\Omega)\colon f = 0 \text{ on } X \cap \Omega\}$. Suppose that $\mathscr{I}_X$ is generated by finitely many holomorphic functions $f^1, \ldots, f^q$ in some neighborhood U of $\overline{\Omega}$. (This is always true if Ω is bounded.) Clearly, $\mathscr{A}^\infty(\Omega; X) \supset ((f) \cdot \mathscr{A}^\infty(\Omega))^\wedge$, where $(f) = (f^1, \ldots, f^q)$ denotes the ideal in $\mathscr{O}(U)$ generated by the f^i, and $((f) \cdot \mathscr{A}^\infty(\Omega))^\wedge$ is the ideal in $\mathscr{A}^\infty(\Omega)$ of elements which formally belong to $(f) \cdot \mathscr{A}^\infty(\Omega)$, as before. Theorem 1.2 gives natural sufficient conditions under which $\mathscr{A}^\infty(\Omega; X) = (f) \cdot \mathscr{A}^\infty(\Omega)$, provided that $\mathscr{A}^\infty(\Omega; X) = ((f) \cdot \mathscr{A}^\infty(\Omega))^\wedge$.

THEOREM 1.4. *Assume that, for each $k = 0, 1, \ldots,$ the closure of $(\Sigma_k(\mathscr{I}_X) - \Sigma_{k+1}(\mathscr{I}_X)) \cap \Omega$ contains $(\Sigma_k(\mathscr{I}_X) - \Sigma_{k+1}(\mathscr{I}_X)) \cap \overline{\Omega}$. Then*

$$\mathscr{A}^\infty(\Omega; X) = \left((f^1, \ldots, f^q) \cdot \mathscr{A}^\infty(\Omega)\right)^\wedge.$$

Theorem 1.4, proved in §10, is another application of semicontinuity of the invariant diagram $\mathfrak{N}_a$ and the formal division algorithm, which is recalled in §3.

Our $\mathscr{C}^\infty$ division theorem is proved in §6. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let U be an open subset of $\mathbf{K}^n$ and let X be a closed subset of U. Let $\mathscr{E}(X)$ denote the Fréchet space of $\mathbf{K}$-valued $\mathscr{C}^\infty$ Whitney functions on X (cf. §6). If Ω is a domain with smooth boundary, then $\mathscr{E}(\overline{\Omega})$ identifies with the space of $\mathbf{K}$-valued $\mathscr{C}^\infty$ functions on $\overline{\Omega}$.

Let $\mathscr{O}$ denote the sheaf of germs of analytic functions on U. Let $a \in U$ and let $\hat{\mathscr{O}}_a$ denote the completion of $\mathscr{O}_a$ in the Krull topology. Put $\hat{\mathscr{O}}_a^{\mathbf{R}} = \hat{\mathscr{O}}_a$ if $\mathbf{K} = \mathbf{R}$, and $\hat{\mathscr{O}}_a^{\mathbf{R}} = \mathbf{C}[[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]]$ if $\mathbf{K} = \mathbf{C}$. If $a \in X$, there is a Taylor series homomorphism $F \mapsto \hat{F}_a$ of $\mathscr{E}(X)^p$ onto $(\hat{\mathscr{O}}_a^{\mathbf{R}})^p$ (onto by E. Borel's lemma [**22**, IV.3.5]). If

M is a submodule of $\mathscr{E}(X)^p$, put $\hat{M} = \{F \in \mathscr{E}(X)^p\colon \hat{F}_a \in \hat{M}_a, \text{ for all } a \in X\}$; clearly, $\hat{M}$ is a closed submodule of $\mathscr{E}(X)^p$ containing M.

THEOREM 1.5. *Let A be a $p \times q$ matrix with entries in $\mathcal{O}(U)$, and let $\mathscr{F} \subset \mathcal{O}^p$ denote the sheaf of $\mathcal{O}$-modules generated by the columns of A. Let $U = \Sigma_0(\mathscr{F}) \supset \Sigma_1(\mathscr{F}) \supset \cdots$ denote the filtration of U by closed analytic subsets determined by the invariant diagram $\mathfrak{G}(\hat{\mathscr{F}}_a)$. If each $\Sigma_k(\mathscr{F})$ is regularly situated with respect to X, then*

$$A \cdot \mathscr{E}(X)^q = \left(A \cdot \mathscr{E}(X)^q\right)^\wedge;$$

in particular, $A \cdot \mathscr{E}(X)^q$ is a closed submodule of $\mathscr{E}(X)^p$.

The regular situation hypotheses are always satisfied if X is subanalytic (cf. [**4**, Theorem 0.1.1]). The conclusion of Theorem 1.5 in the case that X is analytic is Malgrange's division theorem [**17**, Chapter VI]. The techniques we use to deduce Theorem 1.5 (cf. Theorem 6.4) from Malgrange's theorem can be used to prove the latter with little extra effort, so we include it for completeness (Lemma 6.7; cf. [**3**, §10]). The $\mathscr{A}^\infty$ division theorem 1.2 is deduced from Theorem 1.5 in §8 (cf. [**20**]). The regular situation hypotheses of Theorem 1.5 are about as good as can be expected for arbitrary closed sets X; however, the examples of §7 suggest that weaker hypotheses might suffice for domains with smooth boundary.[2]

2. The diagram of initial exponents. Let K be a ring and let $K[[y]]$ denote the ring of formal power series in $y = (y_1, \ldots, y_n)$ with coefficients in K. Following Hironaka [**5**] (cf. [**3**, §1.4]), we associate to every submodule R of $K[[y]]^p$ a subset $\mathfrak{N}(R)$ of $\mathbf{N}^n \times \{1, \ldots, p\}$:

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, put $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The lexicographic ordering of $(n+2)$-tuples $(|\alpha|, j, \alpha_1, \ldots, \alpha_n)$, where $(\alpha, j) \in \mathbf{N}^n \times \{1, \ldots, p\}$, induces a total ordering of $\mathbf{N}^n \times \{1, \ldots, p\}$. Let $f \in K[[y]]^p$. Write $f = \Sigma_{\alpha,j} f_{\alpha,j} y^{\alpha,j}$, where $f_{\alpha,j} \in K$ and $y^{\alpha,j}$ denotes the p-tuple $(0, \ldots, y^\alpha, \ldots, 0)$ with y^α in the jth place. Let $\operatorname{supp} f = \{(\alpha, j) \in \mathbf{N}^n \times \{1, \ldots, p\}\colon f_{\alpha,j} \neq 0\}$ and let $\nu(f)$ denote the smallest element of $\operatorname{supp} f$. Let $\operatorname{in} f$ denote $f_{\nu(f)} y^{\nu(f)}$.

Let R be a submodule of $K[[y]]^p$. We define the *diagram of initial exponents* $\mathfrak{N}(R)$ as $\{\nu(f)\colon f \in R\}$. Clearly, $\mathfrak{N}(R) + \mathbf{N}^n = \mathfrak{N}(R)$, where addition is defined by $(\alpha, j) + \gamma = (\alpha + \gamma; j)$, $(\alpha, j) \in \mathbf{N}^n \times \{1, \ldots, p\}$, $\gamma \in \mathbf{N}^n$.

Put $\mathscr{D}(n, p) = \{\mathfrak{N} \subset \mathbf{N}^n \times \{1, \ldots, p\}\colon \mathfrak{N} + \mathbf{N}^n = \mathfrak{N}\}$. Let $\mathfrak{N} \in \mathscr{D}(n, p)$. Then there is a smallest finite subset $\mathfrak{V}$ of $\mathfrak{N}$ such that $\mathfrak{N} = \mathfrak{V} + \mathbf{N}^n$. We call $\mathfrak{V}$ the *vertices* of $\mathfrak{N}$.

The set $\mathscr{D}(n, p)$ is totally ordered as follows: To each $\mathfrak{N} \in \mathscr{D}(n, p)$, associate the sequence $v(\mathfrak{N})$ obtained by listing the vertices of $\mathfrak{N}$ in ascending order and completing this list to an infinite sequence by using ∞ for all the remaining terms. If $\mathfrak{N}^1, \mathfrak{N}^2 \in \mathscr{D}(n, p)$, we say that $\mathfrak{N}^1 < \mathfrak{N}^2$ provided that $v(\mathfrak{N}^1) < v(\mathfrak{N}^2)$ with respect to the lexicographic ordering on the set of such sequences.

Clearly, if $\mathfrak{N}^1 \supset \mathfrak{N}^2$, then $\mathfrak{N}^1 \leqslant \mathfrak{N}^2$.

[2] Further evidence is provided by the dissertation of M. Hickel, *Quelques résultats de division dans l'algèbre* $\mathscr{A}^\infty(\Omega)$ (Université de Bordeaux I, 1986): In the case that $p = 1$ and X is a domain with smooth boundary, Hickel gives a sufficient condition which covers the examples of §7.

3. The formal division algorithm. We use the notation of §2. Suppose K is a field $\mathbf{K}$. Let $f^1, \ldots, f^s \in \mathbf{K}[[y]]^p$ and let $(\alpha_i, j_i) = \nu(f^i)$, $i = 1, \ldots, s$. We associate to $f^1, \ldots, f^s$ the following decomposition of $\mathbf{N}^n \times \{1, \ldots, p\}$: Set $\Delta_0 = \varnothing$ and define $\Delta_i = ((\alpha_i, j_i) + \mathbf{N}^n) - \bigcup_{k=0}^{i-1} \Delta_k$, $i = 1, \ldots, s$. Put $\Delta = \mathbf{N}^n \times \{1, \ldots, p\} - \bigcup_{i=1}^{s} \Delta_i$.

THEOREM 3.1 (HIRONAKA [5]; CF. [3, §6]). *Let $f^1, \ldots, f^s \in \mathbf{K}[[y]]^p$ and let $(\alpha_i, j_i) = \nu(f^i)$, $i = 1, \ldots, s$. Then, for every $f \in \mathbf{K}[[y]]^p$, there exist unique $q_i \in \mathbf{K}[[y]]$, $i = 1, \ldots, s$, and $r \in \mathbf{K}[[y]]^p$ such that $(\alpha_i, j_i) + \operatorname{supp} q_i \subset \Delta_i$, $i = 1, \ldots, s$, $\operatorname{supp} r \subset \Delta$, and*

$$f = \sum_{i=1}^{s} q_i \cdot f^i + r.$$

REMARK 3.2. Let B be an integral domain. Suppose that $\mathbf{K}$ is the field of fractions of B. Then $B[[y]]$ is a subring of $\mathbf{K}[[y]]$. Suppose that $f^1, \ldots, f^s \in B[[y]]^p$. Let S denote the multiplicative subset of B generated by the $f^i_{\alpha_i, j_i}$, and let $S^{-1}B$ denote the subring of $\mathbf{K}$ comprising quotients with denominators in S. Then $S^{-1}B[[y]] \subset \mathbf{K}[[y]]$. In Theorem 3.1, if $f \in B[[y]]^p$, then $q_i \in S^{-1}B[[y]]$, $i = 1, \ldots, s$, and $r \in S^{-1}B[[y]]^p$ (cf. [3, Remark 6.5]). In fact, if B is any ring and each $f^i_{\alpha_i, j_i} = 1$, the formal division algorithm applies to give quotients and remainder with coefficients in B.

REMARK 3.3. Suppose that $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and that f and $f^1, \ldots, f^s$ all converge. Then the q_i and r all converge [5].

COROLLARY 3.4 (CF. [3, COROLLARY 6.8]). *Let R be a submodule of $\mathbf{K}[[y]]^p$. Let $\mathfrak{N} = \mathfrak{N}(R)$ be the diagram of initial exponents of R, and let (α_i, j_i), $i = 1, \ldots, s$, denote the vertices of $\mathfrak{N}$ (without repetitions). Choose $f^i \in R$ such that $\nu(f^i) = (\alpha_i, j_i)$, $i = 1, \ldots, s$. Then*:

(1) $\mathfrak{N} = \bigcup_{i=1}^{s} \Delta_i$, *and $f^1, \ldots, f^s$ generate R.*

(2) *There is a unique set of generators $g^1, \ldots, g^s$ of R such that, for each i, $\operatorname{in} g^i = y^{\alpha_i, j_i}$ and $\operatorname{supp}(g^i - y^{\alpha_i, j_i}) \cap \mathfrak{N} = \varnothing$. If $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and R is generated by convergent elements, then each g^i converges.*

We call $g^1, \ldots, g^s$ in (2) the *standard basis* of R.

4. Power series with meromorphic coefficients. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let U be an open subset of $\mathbf{K}^m$ and let X be a closed analytic subset of U. Let Z be a proper closed analytic subset of X.

Let $\mathscr{M}(X; Z)$ denote the ring of *meromorphic functions on X whose poles lie in Z*: If $\mathbf{K} = \mathbf{C}$, then $h \in \mathscr{M}(X; Z)$ if h is holomorphic on $X - Z$ and, for every $a \in X$, there exist $f, g \in \mathscr{O}_{X_a}$ such that g is not a zero divisor and $h \cdot g = f$ on (the germ at a of) $X - Z$ (cf. [21, Chapter IV, §5]). (X_a denotes the germ of X at a, and $\mathscr{O}_{X_a}$ the local ring of X_a.) If $\mathbf{K} = \mathbf{R}$, a function on $X - Z$ belongs to $\mathscr{M}(X; Z)$ if its germ at each $a \in X$ is induced by a complex meromorphic function on a complexification $X_a^{\mathbf{C}}$ of X_a whose poles do not intersect $X - Z$.

If $Z = \varnothing$, then $\mathscr{M}(X; Z)$ is the ring $\mathscr{O}(X)$ of analytic functions on X.

LEMMA 4.1. *Let h be a function on $X - Z$. Then $h \in \mathscr{M}(X; Z)$ if and only if each point of X admits a neighborhood in which there are finitely many pairs of analytic functions f_i, g_i such that $X \cap \bigcap_i g_i^{-1}(0) \subset Z$ and $h \cdot g_i = f_i$ on $X - Z$, for each i.*

LEMMA 4.2. *Assume that* $\mathbf{K} = \mathbf{C}$. *Suppose that* $X = \bigcup_{j=1}^k X_j$, *where each* X_j *is a closed analytic subset of* U. *Put* $Z_j = Z \cap X_j$, $j = 1, \dots, k$. *Let* h *be an analytic function on* $X - Z$. *Then* $h \in \mathscr{M}(X; Z)$ *if and only if* $h|(X_j - Z_j) \in \mathscr{M}(X_j; Z_j)$, $j = 1, \dots, k$.

Proofs of the lemmas above are standard. Lemma 4.2 is false in the real case. For example, define $X \subset \mathbf{R}^3$ by $X = X_1 \cup X_2$, where X_1 and X_2 are the zero sets of $z^3 - x^2y^3$ and $z^3 + x^2y^3$, respectively. Let $h = z + x^{2/3}y$. Then h is analytic on $X - Z$, where Z is $\{x = z = 0\}$. Clearly, $h = 2z$ on $X_1 - Z$ and $h = 0$ on $X_2 - Z$; in particular, $h|(X_1 - Z) \in \mathscr{M}(X_1; Z)$ and $h|(X_2 - Z) \in \mathscr{M}(X_2; Z)$. But $h \notin \mathscr{M}(X; Z)$.

Let $a \in X - Z$. There is an evaluation mapping $h \mapsto h(a)$ of $\mathscr{M}(X; Z)$ onto $\mathbf{K}$. Let $y = (y_1, \dots, y_n)$. If $f = \sum f_{\alpha,j} y^{\alpha,j} \in \mathscr{M}(X; Z)[[y]]^p$, we write $f(x; y) = \sum f_{\alpha,j}(x) y^{\alpha,j}$ and $f(a; y) = \sum f_{\alpha,j}(a) y^{\alpha,j}$ when the coefficients are evaluated at $x = a$.

If $f^1, \dots, f^q \in \mathscr{M}(X; Z)[[y]]^p$, we can consider the parametrized family of submodules $\mathscr{R}_a$ of $\mathbf{K}[[y]]^p$, $a \in X - Z$, where each $\mathscr{R}_a$ is generated by $f^i(a; y)$, $i = 1, \dots, q$.

EXAMPLES 4.3. (1) Let V be an open subset of $\mathbf{K}^n$ and let $\mathscr{O} = \mathscr{O}_V$ denote the sheaf of germs of analytic functions on V. Let $\mathscr{F} \subset \mathscr{O}^p$ be a coherent sheaf of $\mathscr{O}$-modules. Suppose that $f_1, \dots, f_q \in \mathscr{O}(V)^p$ generate $\mathscr{F}_a$, for all $a \in V$. For each $i = 1, \dots, q$, $f_i(x + y) = \sum_\alpha D^\alpha f_i(x) \cdot y^\alpha/\alpha!$, where $D^\alpha = \partial^{|\alpha|}/\partial y_1^{\alpha_1} \cdots \partial y_n^{\alpha_n}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. Then $f^i(x; y) = f_i(x + y) \in \mathscr{O}(V)[[y]]^p$ and the $f^i(a; y)$ generate $\hat{\mathscr{F}}_a$, for all $a \in V$. ($\hat{\mathscr{F}}_a \subset \mathbf{K}[[y]]^p$ denotes the completion of $\mathscr{F}_a$.)

(2) Let $M(n, \mathbf{K})$ denote the space of $n \times n$ matrices with entries in $\mathbf{K}$. Then $M(n, \mathbf{K})$ operates on $\mathbf{K}[[y]]^p$ by transformation of the coordinates: If $\lambda \in M(n, \mathbf{K})$ and $f \in \mathbf{K}[[y]]^p$, write $f^\lambda(y) = f(\lambda \cdot y)$. Let R be a submodule of $\mathbf{K}[[y]]^p$. For each λ, let R^λ denote the submodule of $\mathbf{K}[[y]]^p$ generated by $\{f^\lambda\colon f \in R\}$. If $\lambda \in GL(n, \mathbf{K})$, then $R^\lambda = \{f^\lambda\colon f \in R\}$. Suppose that $f_1, \dots, f_q$ generate R. Then each $f^i(\lambda; y) = f_i^\lambda(y)$ is a p-tuple of formal power series in y whose coefficients are homogeneous polynomials in $\lambda \in M(n, \mathbf{K}) = \mathbf{K}^{n^2}$, and the $f^i(\lambda; y)$ generate R^λ, for all $\lambda \in M(n, \mathbf{K})$.

THEOREM 4.4. *Let* $f^1, \dots, f^q \in \mathscr{M}(X; Z)[[y]]^p$. *For each* $a \in X - Z$, *let* $\mathscr{R}_a$ *denote the submodule of* $\mathbf{K}[[y]]^p$ *generated by the* $f^i(a; y)$, $i = 1, \dots, q$, *and put* $\mathfrak{N}_a = \mathfrak{N}(\mathscr{R}_a)$. *Then*:

(1) *For any compact subset* K *of* X, *there are only finitely many distinct* $\mathfrak{N}_a$, $a \in K \cap (X - Z)$.

(2) *For each* $a_0 \in X - Z$, $Z \cup \{a \in X - Z\colon \mathfrak{N}_a \geq \mathfrak{N}_{a_0}\}$ *is a closed analytic subset of* X.

If $Z = \varnothing$ and the conditions (1), (2) of Theorem 4.4 hold, then $\mathfrak{N}_a$ is upper semicontinuous in the *analytic Zariski topology* of X (*Zariski semicontinuous* on X, for short).

PROOF OF THEOREM 4.4. Let $\mathscr{R}$ denote the submodule of $\mathscr{M}(X; Z)[[y]]^p$ generated over $\mathscr{M}(X; Z)[[y]]$ by $f^1, \dots, f^q$. Put $\mathfrak{N} = \mathfrak{N}(\mathscr{R})$ (cf. §2). It suffices to prove the following two lemmas:

LEMMA 4.5 [3, LEMMA 7.1]. *For all $a \in X - Z$, $\mathfrak{N} \leqslant \mathfrak{N}_a$.*

LEMMA 4.6 [3, LEMMA 7.2]. *Assume that $0 \in X$, that the germ of X at 0 is irreducible, and that every connected component of the smooth points of X is adherent to 0. Then there is a proper analytic subset Y of X containing Z such that $\mathfrak{N}_a = \mathfrak{N}$, for all $a \in X - Y$. In fact, for every vertex (α, j) of $\mathfrak{N}$, there exists $g \in \mathscr{R}$ such that $\nu(g) = (\alpha, j) = \nu(g(a; \cdot))$, for all $a \in X - Y$.*

PROOF OF LEMMA 4.5. Let $a \in X - Z$. Let (α_i, j_i), $i = 1, \dots, s$ (respectively, (β_i, k_i), $i = 1, \dots, t$) denote the vertices of $\mathfrak{N}_a$ (respectively, $\mathfrak{N}$) indexed in ascending order.

Consider $h \in \mathscr{R}_a$ such that $\nu(h) = (\alpha_1, j_1)$. Say $h(y) = \sum_{l=1}^q c_l(y) f^l(a; y)$, $c_l(y) \in \mathbf{K}[[y]]$. Define $g \in \mathscr{R}$ by $g(x; y) = \sum c_l(y) f^l(x; y)$. Then $\nu(g) \leqslant (\alpha_1, j_1)$ since, in any case, the coefficient of y^{α_1, j_1} is nonzero. Thus $(\beta_1, k_1) \leqslant \nu(g) \leqslant (\alpha_1, j_1)$. If $(\beta_1, k_1) = (\alpha_1, j_1)$, then $\nu(g) = (\alpha_1, j_1) = \nu(g(a; \cdot))$.

Now suppose that, for each $i = 1, \dots, r$, we have: (i) $(\beta_i, k_i) = (\alpha_i, j_i)$, and (ii) there exists $g^i(x; y) \in \mathscr{R}$ such that $\nu(g^i) = (\alpha_i, j_i) = \nu(g^i(a; \cdot))$. If $s = r$, we are done. Otherwise, consider $h(y) = \sum c_l(y) f^l(a; y) \in \mathscr{R}_a$ such that $\nu(h) = (\alpha_{r+1}, j_{r+1})$; say in $h = y^{\alpha_{r+1}, j_{r+1}}$. Put $g(x; y) = \sum c_l(y) f^l(x; y) \in \mathscr{R}$. Then $\nu(g) \leqslant (\alpha_{r+1}, j_{r+1})$. If $\nu(g) = (\alpha_{r+1}, j_{r+1})$, then $(\beta_{r+1}, k_{r+1}) \leqslant (\alpha_{r+1}, j_{r+1})$. If $\nu(g) < (\alpha_{r+1}, j_{r+1})$, then either $\nu(g) \notin \bigcup_{i=1}^r ((\alpha_i, j_i) + \mathbf{N}^n)$ and $(\beta_{r+1}, k_{r+1}) < (\alpha_{r+1}, j_{r+1})$, or $\nu(g) \in \bigcup_{i=1}^r ((\alpha_i, j_i) + \mathbf{N}^n)$. In the latter case, $\nu(g) = (\alpha_i + \gamma, j_i)$ for some $i = 1, \dots, r$ and some $\gamma \in \mathbf{N}^n$. Then in $g = g_{\alpha_i + \gamma, j_i}(x) \cdot y^{\alpha_i + \gamma, j_i}$, where $g_{\alpha_i + \gamma, j_i}(a) = 0$ since $\nu(g) < (\alpha_{r+1}, j_{r+1}) = \nu(g(a; \cdot))$. On the other hand, in $g^i = g^i_{\alpha_i, j_i}(x) \cdot y^{\alpha_i, j_i}$, where $g^i_{\alpha_i, j_i}(a) \neq 0$. Let

$$g'(x; y) = g^i_{\alpha_i, j_i}(x) \cdot g(x; y) - g_{\alpha_i + \gamma, j_i}(x) \cdot y^\gamma \cdot g^i(x; y).$$

Then $\nu(g'(a; \cdot)) = (\alpha_{r+1}, j_{r+1})$ and $\nu(g) < \nu(g') \leqslant (\alpha_{r+1}, j_{r+1})$. The result follows by induction. □

PROOF OF LEMMA 4.6. Let (β_i, k_i), $i = 1, \dots, t$, denote the vertices of $\mathfrak{N}$. For each i, choose $g^i \in \mathscr{R}$ such that $\nu(g^i) = (\beta_i, k_i)$; write $g^i(x; y) = \sum g^i_{\alpha, j}(x) \cdot y^{\alpha, j}$. Put

$$Y = Z \cup \bigcup_{i=1}^t \left\{ x \in X : g^i_{\beta_i, k_i}(x) = 0 \right\}.$$

Let $a \in X - Y$. Then $g^i(a; y) \in \mathscr{R}_a$ and $\nu(g^i(a; \cdot)) = (\beta_i, k_i)$. Thus $\mathfrak{N} \subset \mathfrak{N}_a$. By Lemma 4.5, $\mathfrak{N}_a = \mathfrak{N}$. □

COROLLARY 4.7. *Let U be an open subset of $\mathbf{K}^n$ and let $\mathscr{F} \subset \mathscr{O}^p$ be a coherent sheaf of $\mathscr{O}$-modules, where $\mathscr{O} = \mathscr{O}_U$. Then*:

(1) *The diagram of initial exponents $\mathfrak{N}_a = \mathfrak{N}(\hat{\mathscr{F}}_a)$ is Zariski semicontinuous on U. Thus, there is a locally finite filtration of U by closed analytic subsets, $U = \Sigma_0(\mathscr{F}) \supset \Sigma_1(\mathscr{F}) \supset \cdots$, such that, for each $k = 0, 1, \dots,$*

(i) *$\mathfrak{N}_a$ is constant, say $\mathfrak{N}_a = \mathfrak{N}_k(\mathscr{F})$, on $\Sigma_k(\mathscr{F}) - \Sigma_{k+1}(\mathscr{F})$;*

(ii) *$\mathfrak{N}_k(\mathscr{F}) < \mathfrak{N}_{k+1}(\mathscr{F})$.*

(2) *Let $k \in \mathbf{N}$ and let (α_i, j_i), $i = 1, \ldots, s$, denote the vertices of $\mathfrak{N}_k(\mathscr{F})$.* Let f_a^i, $i = 1, \ldots, s$, *denote the standard basis of $\hat{\mathscr{F}}_a \subset \mathbf{K}[[y]]^p$, where $a \in \Sigma_k(\mathscr{F}) - \Sigma_{k+1}(\mathscr{F})$ and* in $f_a^i = y^{\alpha_i, j_i}$. *Write $f_a^i(y) = \sum f_{\alpha,j}^i(a) y^{\alpha,j}$. Then*:

(i) *There exist p-tuples of analytic functions f^i defined in a neighborhood of $\Sigma_k(\mathscr{F}) - \Sigma_{k+1}(\mathscr{F})$, whose power series expansions at each $a \in \Sigma_k(\mathscr{F}) - \Sigma_{k+1}(\mathscr{F})$ are the f_a^i.*

(ii) *Each $f_{\alpha,j}^i \in \mathscr{M}(\Sigma_k(\mathscr{F}); \Sigma_{k+1}(\mathscr{F}))$.*

PROOF. (1) is a special case of Theorem 4.4 (cf. Example 4.3(1)). For (2)(i): Each f_a^i converges, by Corollary 3.4(2). For b in a sufficiently small neighborhood of a in $\Sigma_k(\mathscr{F}) - \Sigma_{k+1}(\mathscr{F})$, $f_a^i(b - a + y) \in \mathscr{F}_b$. Now,

$$\operatorname{supp}\left(f_a^i(b - a + y) - y^{\alpha_i, j_i}\right) \cap \mathfrak{N}_b = \varnothing,$$

since $\mathfrak{N}_a = \mathfrak{N}_b$ and $\operatorname{supp}(f_a^i(y) - y^{\alpha_i, j_i}) \cap \mathfrak{N}_a = \varnothing$. Hence in $f_a^i(b - a + y) = y^{\alpha_i, j_i}$ and, by the uniqueness of formal division (Theorem 3.1), $f_a^i(b - a + y) = f_b^i(y)$. Finally, since Corollary 4.7 in the real case follows from the complex case, (2)(ii) follows from Remark 3.2 and Lemma 4.2. □

5. The invariant diagram of initial exponents. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let R be a submodule of $\mathbf{K}[[y]]^p$, $y = y(y_1, \ldots, y_n)$. The diagram of initial exponents $\mathfrak{N}(R)$ depends on the affine coordinate system $(y_1, \ldots, y_n)$. However, there is an invariant diagram $\mathfrak{G}(R)$ (cf. [**9,11**]):

Suppose $\lambda = \lambda(y)$ is a formal coordinate transformation. If $f \in \mathbf{K}[[y]]^p$, write $f^\lambda(y) = f(\lambda(y))$. Put $R^\lambda = \{f^\lambda\colon f \in R\}$. Then R^λ is a submodule of $\mathbf{K}[[y]]^p$. Clearly, $\mathfrak{N}(R^\lambda)$ depends only on the linear part of λ, so to study the effect of coordinate changes on $\mathfrak{N}(R)$, we can assume that $\lambda \in GL(n, \mathbf{K})$.

We view $GL(n, \mathbf{K})$ as a subset of the space $M(n, \mathbf{K}) = \mathbf{K}^{n^2}$ of $n \times n$ matrices with entries in $\mathbf{K}$. Of course, $\mathbf{K}^{n^2} - GL(n, \mathbf{K})$ is a closed algebraic subset of $\mathbf{K}^{n^2}$. For any $\lambda \in M(n, \mathbf{K})$, let R^λ denote the submodule of $\mathbf{K}[[y]]^p$ generated by $\{f^\lambda\colon f \in R\}$ (cf. Example 4.3(2)). The following is a special case of Theorem 4.4:

PROPOSITION 5.1. *Let R be a submodule of $\mathbf{K}[[y]]^p$. Then $\mathfrak{N}(R^\lambda)$ is Zariski semicontinuous on $M(n, \mathbf{K})$.*

The proof of Theorem 4.4 shows, in fact, that $\mathfrak{N}(R^\lambda)$ is upper semicontinuous in the (algebraic) Zariski topology of $M(n, \mathbf{K})$. In particular, there are only finitely many distinct $\mathfrak{N}(R^\lambda)$, for $\lambda \in M(n, \mathbf{K})$.

We define the *invariant diagram of initial exponents* $\mathfrak{G}(R)$ as $\min_\lambda \mathfrak{N}(R^\lambda)$. Then $\{\lambda \in M(n, \mathbf{K})\colon \mathfrak{N}(R^\lambda) = \mathfrak{G}(R)\}$ is a Zariski open subset. Obviously, $\mathfrak{G}(R) = \min_{\lambda \in GL(n, \mathbf{K})} \mathfrak{N}(R^\lambda)$.

PROPOSITION 5.2. *Let U be an open subset of $\mathbf{K}^n$ and let $\mathscr{F} \subset \mathscr{O}^p$ be a coherent sheaf of $\mathscr{O}$-modules, where $\mathscr{O} = \mathscr{O}_U$. Then $\mathfrak{G}_a = \mathfrak{G}(\hat{\mathscr{F}}_a)$ is Zariski semicontinuous on U.*

We can prove this in the more general context of Theorem 4.4:

THEOREM 5.3. *Let X, Z, f^i, $\mathscr{R}_a$, etc. be as in Theorem 4.4. For each $a \in X - Z$, put $\mathfrak{G}_a = \mathfrak{G}(\mathscr{R}_a)$. Then*

(1) *For any compact subset K of X, there are only finitely many distinct $\mathfrak{G}_a$, $a \in K \cap (X - Z)$.*

(2) *For each* $a_0 \in X - Z$, $Z \cup \{a \in X - Z\colon \mathfrak{G}_a \geqslant \mathfrak{G}_{a_0}\}$ *is a closed analytic subset of* X.

PROOF. If $a \in X - Z$ and $\lambda \in M(n, \mathbf{K})$, then $\mathscr{R}_a^\lambda$ is generated by the $f^i(a; \lambda \cdot y)$, $i = 1, \ldots, q$. Clearly, each $f^i(x; \lambda \cdot y) \in \mathscr{M}(X \times \mathbf{K}^{n^2}; Z \times \mathbf{K}^{n^2})[[y]]^p$. Let K be a compact subset of X and let L be a compact neighborhood of 0 in $\mathbf{K}^{n^2}$. By Theorem 4.4(1), there are only finitely many distinct $\mathfrak{N}(\mathscr{R}_a^\lambda)$, for $(a, \lambda) \in (K \times L) \cap ((X - Z) \times \mathbf{K}^{n^2})$. But, for each a, $\mathfrak{G}_a = \min_{\lambda \in L} \mathfrak{N}(\mathscr{R}_a^\lambda)$. Therefore, there are only finitely many distinct $\mathfrak{G}_a$, for $a \in K \cap (X - Z)$.

Suppose $a_0 \in X - Z$. By Theorem 4.4(2), $Y = (Z \times \mathbf{K}^{n^2}) \cup \{(a, \lambda) \in (X - Z) \times \mathbf{K}^{n^2}\colon \mathfrak{N}(\mathscr{R}_a^\lambda) \geqslant \mathfrak{G}_{a_0}\}$ is a closed analytic subset of $X \times \mathbf{K}^{n^2}$. For each $\lambda \in \mathbf{K}^{n^2}$, $Y_\lambda = \{a \in X\colon (a, \lambda) \in Y\}$ is a closed analytic subset of X. Clearly, $Z \cup \{a \in X - Z\colon \mathfrak{G}_a \geqslant \mathfrak{G}_{a_0}\} = \bigcap_{\lambda \in \mathbf{K}^{n^2}} Y_\lambda$. □

REMARK 5.4. The invariant diagram is interesting in a much broader context than that of Theorem 5.3. In [**3**], we consider the diagram of initial exponents in the following general setting: Let X and Y be analytic spaces over $\mathbf{K}$, and let $\phi: X \to Y$ be a morphism. Let $\phi^*\colon \mathscr{O}_Y \to \mathscr{O}_X$ denote the induced homomorphism of the structure sheaves. Suppose that $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_X$- and $\mathscr{O}_Y$-modules, respectively, and that $\Psi\colon \mathscr{G} \to \mathscr{F}$ is a module homomorphism over the ring homomorphism ϕ^*. Let $a \in X$. Then ϕ^* determines a homomorphism of local rings $\phi_a^*\colon \mathscr{O}_{Y, \phi(a)} \to \mathscr{O}_{X, a}$ and Ψ determines a module homomorphism $\Psi_a\colon \mathscr{G}_{\phi(a)} \to \mathscr{F}_a$ over ϕ_a^*. Let $\hat{\Psi}_a\colon \hat{\mathscr{G}}_{\phi(a)} \to \hat{\mathscr{F}}_a$ denote the induced homomorphism of the completions.

Let $\mathscr{R}_a$ denote the "module of formal relations" $\mathscr{R}_a = \operatorname{Ker} \hat{\Psi}_a$. To study the variation of $\mathscr{R}_a$ with respect to a, we reduce to the case that Y is smooth and $\mathscr{G} = \mathscr{O}_Y^q$. Then we can consider the diagram $\mathfrak{N}_a = \mathfrak{N}(\mathscr{R}_a)$ relative to a local coordinate system $(y_1, \ldots, y_n)$ for Y, or the diagram $\mathfrak{G}_a = \mathfrak{G}(\mathscr{R}_a)$ invariantly.

Suppose that $\mathfrak{N}_a$ (relative to local coordinates for Y) is Zariski semicontinuous as a function of $a \in X$. Then there is a locally finite filtration of X by closed analytic subsets, $X = \Sigma_0 \supset \Sigma_1 \supset \cdots$, such that, for all $k = 0, 1, \ldots$, (1) $\mathfrak{N}_a$ is constant, say $\mathfrak{N}_a = \mathfrak{N}_k$, on $\Sigma_k - \Sigma_{k+1}$, and (2) $\mathfrak{N}_k < \mathfrak{N}_{k+1}$. Let $k \in \mathbf{N}$ and let g_a^i, $i = 1, \ldots, t$, denote the standard basis of $\mathscr{R}_a \subset \mathbf{K}[[y]]^q$, where $a \in \Sigma_k - \Sigma_{k+1}$. Write $g_a^i(y) = \sum_{\beta, j} g_{\beta, j}^i(a) y^{\beta, j}$. Then each $g_{\beta, j}^i \in \mathscr{M}(\Sigma_k; \Sigma_{k+1})$ [**3**, Theorem B]. It follows from Theorem 5.3 that $\mathfrak{G}_a$ is Zariski semicontinuous on X.

In [**3**], we prove that $\mathfrak{N}_a$ is Zariski semicontinuous in many cases; for example, in the algebraic category. Therefore, $\mathfrak{G}_a$ is also Zariski semicontinuous. The standard bases arising, as above, from $\mathfrak{N}_a$ in the algebraic category need not be algebraic. Nevertheless, the standard bases arising from the invariant diagram $\mathfrak{G}_a$ always are, by a Henselian version of the formal division theorem [**13**, §4]. As a consequence, for example, we recover in a very explicit way the stratification of semialgebraic sets by Nash functions [**18**, **23**].

6. $\mathscr{C}^\infty$ division on a closed set. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let U be an open subset of $\mathbf{K}^n$, and let $\mathscr{C}^\infty(U)$ denote the Fréchet algebra of $\mathbf{K}$-valued $\mathscr{C}^\infty$ functions on U, with the $\mathscr{C}^\infty$ topology. Let $(y_1, \ldots, y_n)$ denote the affine coordinates of $\mathbf{K}^n$. If $\mathbf{K} = \mathbf{R}$, set

$m = n$ and $(x_1, \dots, x_m) = (y_1, \dots, y_n)$. If $\mathbf{K} = \mathbf{C}$, put $m = 2n$ and $(x_1, \dots, x_m) = (y_1, \dots, y_n, \bar{y}_1, \dots, \bar{y}_n)$. Then, for each $a \in U$, there is a Taylor series homomorphism $f \mapsto \hat{f}_a$ of $\mathscr{C}^\infty(U)^p$ onto $(\hat{\mathscr{O}}_a^{\mathbf{R}})^p$, where $\hat{\mathscr{O}}_a^{\mathbf{R}} = \mathbf{K}[[x_1, \dots, x_m]]$. Of course, $\hat{\mathscr{O}}_a$ is a subring of $\hat{\mathscr{O}}_a^{\mathbf{R}}$.

Let Y be a locally closed subset of $\mathbf{K}^n$ and let Z be a closed subset of Y (where Y has the induced topology). Let V be any open subset of $\mathbf{K}^n$ such that Y is closed in V. Then Z is closed in V. Put $\mathscr{E}(V; Z) = \{g \in \mathscr{C}^\infty(V)\colon \hat{g}_a = 0, \text{ for all } a \in Z\}$. Let $\mathscr{E}(Y; Z)$ denote the Fréchet algebra $\mathscr{E}(V; Z)/\mathscr{E}(V; Y)$. (The definition is independent of V.) By Whitney's extension theorem [**17**, I.4.1], $\mathscr{E}(Y; Z)$ identifies with the Fréchet algebra of $\mathbf{K}$-valued $\mathscr{C}^\infty$ Whitney "functions" on Y which are flat on Z. If Y' is a closed subset of Y, there is a restriction mapping $F \mapsto F \mid Y'$ from $\mathscr{E}(Y; Z)^p$ onto $\mathscr{E}(Y'; Z \cap Y')^p$. Put $\mathscr{E}(Y) = \mathscr{E}(Y; \varnothing)$. If $a \in Y$, then $\mathscr{E}(\{a\}) = \hat{\mathscr{O}}_a^{\mathbf{R}}$, and we write $F \mapsto \hat{F}_a$ for the restriction $\mathscr{E}(Y)^p \to (\hat{\mathscr{O}}_a^{\mathbf{R}})^p$ (Taylor series homomorphism). Put $J(Y) = \prod_{a \in Y} \hat{\mathscr{O}}_a^{\mathbf{R}}$. There is a canonical inclusion $\mathscr{E}(Y) \hookrightarrow J(Y)$ given by $G \mapsto (\hat{G}_a)_{a \in Y}$. Of course, $\mathscr{E}(V) = \mathscr{C}^\infty(V)$ if V is an open subset of $\mathbf{K}^n$.

We will use the following three lemmas in the proof of Theorem 1.5. Let U be an open subset of $\mathbf{K}^n$.

LEMMA 6.1 ("HESTENES'S LEMMA FOR ANALYTIC SETS"). *Let $Z \subset Y$ be closed subsets of U, where Y is analytic. Let $F \in J(Y)$, say $F = (F_a)$, where $F_a(x) = \sum_{\alpha \in \mathbf{N}^m} f_\alpha(a) x^\alpha \in \hat{\mathscr{O}}_a^{\mathbf{R}}$, $a \in Y$. Suppose that $F \mid (Y - Z) \in \mathscr{E}(Y - Z)$ and that each f_α is continuous on Y and zero on Z. Then $F \in \mathscr{E}(Y)$.*

This can be proved (in fact, for Y merely subanalytic) as in [**2**, Corollary 8.2] or by a direct estimate using the following regularity condition of Whitney: Let K be a compact subset of Y. Then there exist c, $r > 0$ such that any 2 points a, b of K can be joined by a rectifiable arc γ in Y of length $\leqslant c \cdot |a - b|^r$.

LEMMA 6.2. *Let $Z_1, \dots, Z_r$ be closed subsets of U such that, for each $j = 1, \dots, r - 1$, $Z_1 \cap \cdots \cap Z_j$ and Z_{j+1} are regularly situated. Put $Z = \bigcap_{j=1}^r Z_j$. If $F \in \mathscr{E}(U; Z)$, then there exist $F_j \in \mathscr{E}(U; Z_j)$ $j = 1, \dots, r$, such that $F = \sum F_j$.*

PROOF. By Łojasiewicz's gluing theorem [**17**, I.5.5], there exists $F_r \in \mathscr{E}(U; Z_r)$ such that $F_r = F$ on $Z_1 \cap \cdots \cap Z_{r-1}$. Then $F = (F - F_r) + F_r$, where $F - F_r \in \mathscr{E}(U; Z_1 \cap \cdots \cap Z_{r-1})$. Proceed inductively. □

LEMMA 6.3. *Let $Z \subset Y$ be closed analytic subsets of U. Let $\phi \in \mathscr{M}(Y; Z)$ and let f be the restriction to Y of a $\mathscr{C}^\infty$ function which is flat on Z. Then $\phi \cdot f$ extends in a unique way to a function on Y (also denoted $\phi \cdot f$) which is the restriction of a $\mathscr{C}^\infty$ function flat on Z.*

PROOF. By Lemma 4.1, we can assume there are analytic functions ξ_i, η_i, $i = 1, \dots, r$, on U such that $Y \cap \bigcap_{i=1}^r \eta_i^{-1}(0) \subset Z$ and, for each i, $\phi \cdot \eta_i = \xi_i$ on $Y - Z$. Put $P_i = Z \cup \eta_i^{-1}(0)$, $i = 1, \dots, r$, and $P = \bigcap P_i$, so that $P \cap Y = Z$. We can extend f to $F \in \mathscr{E}(U; P)$. By Lemma 6.2, $F = \sum_i F_i$, where each $F_i \in \mathscr{E}(U; P_i)$. Set $\phi_i = \xi_i/\eta_i$, $i = 1, \dots, r$; then each $\phi_i \in \mathscr{M}(U; P_i)$. It follows from Łojasiewicz's inequality [**22**, IV.4.1] that each $\phi_i \cdot F_i$ extends in a unique way to an element of $\mathscr{E}(U; P_i)$. Clearly, $\sum \phi_i \cdot F_i$ restricts to $\phi \cdot f$. □

Let U be an open subset of $\mathbf{K}^n$ and let $\mathcal{O} = \mathcal{O}_U$. Let A denote a $p \times q$ matrix with entries in $\mathcal{O}(U)$, and let $\mathcal{F}$ denote the sheaf of submodules of $\mathcal{O}^p$ generated by the columns $\phi^1, \ldots, \phi^q$ of A. By Proposition 5.2, there is a locally finite filtration of U by closed analytic subsets, $U = \Sigma_0(\mathcal{F}) \supset \Sigma_1(\mathcal{F}) \supset \cdots$, such that, for each $k = 0, 1, \ldots,$

(i) $\mathfrak{G}_a = \mathfrak{G}(\hat{\mathcal{F}}_a)$ is constant, say $\mathfrak{G}_a = \mathfrak{G}_k(\mathcal{F})$, on $\Sigma_k(\mathcal{F}) - \Sigma_{k+1}(\mathcal{F})$;
(ii) $\mathfrak{G}_k(\mathcal{F}) < \mathfrak{G}_{k+1}(\mathcal{F})$.

THEOREM 6.4. *Let X be a closed subset of U. If $\Sigma_k(\mathcal{F})$ and X are regularly situated, for all $k \in \mathbf{N}$, then*

$$A \cdot \mathcal{E}(X)^q = \left(A \cdot \mathcal{E}(X)^q\right)^\wedge.$$

PROOF. It is enough to prove that, for each $k = 0, 1, \ldots,$

$$A \cdot \mathcal{E}(\Sigma_k \cap X;\ \Sigma_{k+1} \cap X)^q = \left(A \cdot \mathcal{E}(\Sigma_k \cap X;\ \Sigma_{k+1} \cap X)^q\right)^\wedge,$$

where $\Sigma_k = \Sigma_k(\mathcal{F})$. Fix k. Let $F \in (A \cdot \mathcal{E}(\Sigma_k \cap X;\ \Sigma_{k+1} \cap X)^q)^\wedge$. Since Σ_{k+1} and X are regularly situated, there exists $F' \in \mathcal{E}(\Sigma_k; \Sigma_{k+1})^p$ such that $F' | \Sigma_k \cap X = F$.

Since $\mathfrak{N}(\hat{\mathcal{F}}_a^\lambda)$ is Zariski semicontinuous on $U \times GL(n, \mathbf{K})$ (by Theorem 4.4), we can assume (working locally in U) that there exist $\lambda_i \in GL(n, \mathbf{K})$ and closed analytic sets Y_i, $\Sigma_{k+1} \subset Y_i \subset \Sigma_k$, $i = 1, \ldots, r$, such that $\bigcap Y_i = \Sigma_{k+1}$ and $\mathfrak{N}(\hat{\mathcal{F}}_a^{\lambda_i}) = \mathfrak{G}(\hat{\mathcal{F}}_a)$, for all $a \in \Sigma_k - Y_i$, $i = 1, \ldots, r$.

There exists $\mu \in \mathcal{E}(\Sigma_k; \Sigma_{k+1})$ such that $\mu > 0$ on $\Sigma_k - \Sigma_{k+1}$ and $F' = \mu \cdot F''$, where $F'' \in \mathcal{E}(\Sigma_k; \Sigma_{k+1})^p$ [**22**, V.2.4]. Clearly, $F'' | \Sigma_k \cap X \in (A \cdot \mathcal{E}(\Sigma_k \cap X)^q)^\wedge$. By Lemma 6.2, there exist $\mu_i \in \mathcal{E}(\Sigma_k; Y_i)$, $i = 1, \ldots, r$, such that $\mu = \sum \mu_i$.

Suppose that, for each $i = 1, \ldots, r$, there exists $G_i \in \mathcal{E}(\Sigma_k; Y_i)^q$ such that $\mu_i \cdot F'' = A \cdot G_i$ on $\Sigma_k \cap X$. Then $G = \sum G_i \in \mathcal{E}(\Sigma_k; \Sigma_{k+1})^q$ and $F = A \cdot G | \Sigma_k \cap X$. Hence it is enough to prove the following proposition.

PROPOSITION 6.5. *Let $Y \subset \Sigma$ be closed analytic subsets of U such that $\mathfrak{N}_a = \mathfrak{N}(\hat{\mathcal{F}}_a)$ is constant on $\Sigma - Y$. Let $F \in \mathcal{E}(\Sigma; Y)^p$. If $F | \Sigma \cap X \in (A \cdot \mathcal{E}(\Sigma \cap X)^q)^\wedge$, then there exists $G \in \mathcal{E}(\Sigma; Y)^q$ such that $F | \Sigma \cap X = A \cdot G | \Sigma \cap X$.*

Let π: $\mathbf{N}^m \times \{1, \ldots, p\} \to \mathbf{N}^n \times \{1, \ldots, p\}$ denote the projection $\pi(\alpha, j) = (\alpha_1, \ldots, \alpha_n, j)$, where $(\alpha, j) \in \mathbf{N}^m \times \{1, \ldots, p\}$, $\alpha = (\alpha_1, \ldots, \alpha_m)$. If $\mathfrak{N} \in \mathcal{D}(n, p)$, put $\mathfrak{N}^{\mathbf{R}} = \pi^{-1}(\mathfrak{N})$; clearly, $\mathfrak{N}^{\mathbf{R}} \in \mathcal{D}(m, p)$. If $a \in U$, then the $\hat{\mathcal{O}}_a$-homomorphism $\hat{A}_a$: $\hat{\mathcal{O}}_a^q \to \hat{\mathcal{O}}_a^p$ induces an $\hat{\mathcal{O}}_a^{\mathbf{R}}$-homomorphism $\hat{A}_a^{\mathbf{R}}$: $(\hat{\mathcal{O}}_a^{\mathbf{R}})^q \to (\hat{\mathcal{O}}_a^{\mathbf{R}})^p$; clearly, $\mathfrak{N}(\operatorname{Im} \hat{A}_a^{\mathbf{R}}) = \mathfrak{N}_a^{\mathbf{R}}$.

PROOF OF PROPOSITION 6.5. Write $\mathfrak{N} = \mathfrak{N}_a$, $a \in \Sigma - Y$. By Theorem 3.1, for all $a \in \Sigma - Y$, there exists a unique $R_a \in (\hat{\mathcal{O}}_a^{\mathbf{R}})^p$ such that $\operatorname{supp} R_a \cap \mathfrak{N}^{\mathbf{R}} = \varnothing$ and $\hat{F}_a - R_a \in \operatorname{Im} \hat{A}_a^{\mathbf{R}}$. In particular, $R_a = 0$ if $a \in (\Sigma - Y) \cap X$. If $a \in Y$, put $R_a = 0$. We will prove the following two lemmas:

LEMMA 6.6. $R = (R_a)_{a \in \Sigma} \in \mathcal{E}(\Sigma; Y)^p$.

LEMMA 6.7. $A \cdot \mathcal{E}(\Sigma; Y)^q = (A \cdot \mathcal{E}(\Sigma; Y)^q)^\wedge$.

Once we have Lemma 6.6, it follows from Lemma 6.7 that there exists $G \in \mathscr{E}(\Sigma; Y)^q$ such that $F = A \cdot G + R$. Since $R | \Sigma \cap X = 0$, this completes the proof of Proposition 6.5. □

Lemma 6.7 is, of course, a special case of Malgrange's division theorem [**17**, Chapter VI]; in fact, it is equivalent to Malgrange's theorem, by Corollary 4.7. But we will give a simple explicit proof using the same techniques we use to prove Lemma 6.6 (cf. [**3**, §10]).

PROOF OF LEMMA 6.6. Let (α_i, j_i), $i = 1, \ldots, s$, denote the vertices of $\mathfrak{N}$. If $a \in \Sigma - Y$, let $\theta_a^i(y)$, $i = 1, \ldots, s$, denote the standard basis of $\hat{\mathscr{F}}_a$. Write $\theta_a^i(y) = \sum_{\alpha, j} \theta_{\alpha, j}^i(a) y^{\alpha, j}$. By Corollary 4.7:

(1) Each $\theta_{\alpha, j}^i \in \mathscr{M}(\Sigma; Y)$.

(2) There are analytic functions θ^i defined in a neighborhood of $\Sigma - Y$ whose power series expansions at each $a \in \Sigma - Y$ are the θ_a^i.

We identify $\mathbf{N}^n$ with $\mathbf{N}^n \times \{0\} \subset \mathbf{N}^m$. Then $\mathfrak{N}^{\mathbf{R}} = \mathfrak{N} + \mathbf{N}^m$ and (α_i, j_i), $i = 1, \ldots, s$, are also the vertices of $\mathfrak{N}^{\mathbf{R}}$. Let $\{\Delta_i, \Delta\}$ denote the decomposition of $\mathbf{N}^m \times \{1, \ldots, p\}$ determined by the (α_i, j_i), as in §3.

Let $a \in \Sigma - Y$. By Theorem 3.1, there exist unique $G_{i,a} \in \hat{\mathscr{O}}_a^{\mathbf{R}}$, $i = 1, \ldots, s$, such that $(\alpha_i, j_i) + \operatorname{supp} G_{i,a} \subset \Delta_i$ and

$$\hat{F}_a = \sum_{i=1}^{s} G_{i,a} \cdot \theta_a^i + R_a. \tag{6.8}$$

If $a \in Y$, put $G_{i,a} = 0$, $i = 1, \ldots, s$. For each $a \in \Sigma$, write $G_{i,a}(x) = \sum_{\alpha \in \mathbf{N}^m} g_{i,\alpha}(a) x^\alpha$, $i = 1, \ldots, s$, and $R_a = \sum_{\alpha, j} r_{\alpha, j}(a) x^{\alpha, j}$.

By Lemma 6.3, Remark 3.2 and (1) above, each $g_{i,\alpha}$ and $r_{\alpha, j}$ is the restriction to Σ of a $\mathscr{C}^\infty$ function which is flat on Y. From Lemma 6.1, we can conclude that each $G_i \in \mathscr{E}(\Sigma; Y)$ and $R \in \mathscr{E}(\Sigma; Y)^p$, provided we show that each $G_i | (\Sigma - Y) \in \mathscr{E}(\Sigma - Y)$ and $R | (\Sigma - Y) \in \mathscr{E}(\Sigma - Y)^p$.

These assertions are local in Σ. Therefore, after perhaps shrinking U, we can assume there is a finite filtration of Σ by closed analytic subsets, $\Sigma = \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_{\nu+1} = Y$, such that, for all $\mu = 0, \ldots, \nu$, $\Sigma_\mu - \Sigma_{\mu+1}$ is smooth. By Lemma 6.1, it is enough to show that, for all $\mu = 0, \ldots, \nu$, each $G_i | (\Sigma_\mu - \Sigma_{\mu+1}) \in \mathscr{E}(\Sigma_\mu - \Sigma_{\mu+1})$ and $R | (\Sigma_\mu - \Sigma_{\mu+1}) \in \mathscr{E}(\Sigma_\mu - \Sigma_{\mu+1})^p$.

Extend F to $f \in \mathscr{E}(U; Y)^p$. Since f is $\mathscr{C}^\infty$ and the θ^i are analytic, then, regarding both a and x as variables, we have

$$\frac{\partial \hat{f}_a(x)}{\partial a_j} = \frac{\partial \hat{f}_a(x)}{\partial x_j}, \quad \frac{\partial \theta_a^i(x)}{\partial a_j} = \frac{\partial \theta_a^i(x)}{\partial x_j}, \tag{6.9}$$

$j = 1, \ldots, m$ ("Taylor expansion commutes with formal differentiation"). If $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m = \mathbf{K}^n$, write $D_{\lambda,a} = \sum \lambda_j \partial/\partial a_j$; $D_{\lambda,a}$ is the directional derivative with respect to the a variables in the direction λ. Let $a \in \Sigma_\mu - \Sigma_{\mu+1}$ and suppose that $D_{\lambda,a}$ is tangent to $\Sigma_\mu - \Sigma_{\mu+1}$ at a. Then the $D_{\lambda,a} G_{i,a}(x)$ and $D_{\lambda,a} R_a(x)$ are well defined. By (6.8) and (6.9),

$$\sum_{i=1}^{s} (D_{\lambda,a} G_{i,a} - D_{\lambda,x} G_{i,a}) \cdot \theta_a^i + D_{\lambda,a} R_a - D_{\lambda,x} R_a = 0.$$

For each i, $(\alpha_i, j_i) + \operatorname{supp}(D_{\lambda,a}G_{i,a} - D_{\lambda,x}G_{i,a}) \subset \Delta_i$, and $\operatorname{supp}(D_{\lambda,a}R_a - D_{\lambda,x}R_a) \subset \Delta$ (where supp is with respect to x). Therefore, by the uniqueness in Theorem 3.1,

$$D_{\lambda,a}R_a = D_{\lambda,x}R_a \tag{6.10}$$

and, for all $i = 1,\ldots,s$, $D_{\lambda,a}G_{i,a} = D_{\lambda,x}G_{i,a}$. These equations imply that $R|(\Sigma_\mu - \Sigma_{\mu+1}) \in \mathscr{E}(\Sigma_\mu - \Sigma_{\mu+1})^p$ and each $G_i|(\Sigma_\mu - \Sigma_{\mu+1}) \in \mathscr{E}(\Sigma_\mu - \Sigma_{\mu+1})$. Here is the argument for R: Choose local coordinates $(u,v) = (u_1,\ldots,u_k,\ v_1,\ldots,v_{m-k})$ near $a \in \Sigma_\mu - \Sigma_{\mu+1}$ such that $\Sigma_\mu - \Sigma_{\mu+1}$ is given by $v = 0$. Write R_a as

$$R_a(u,v) = \sum_{\substack{\beta\in\mathbf{N}^{m-k}\\ j=1,\ldots,p}} \left\{ \sum_{\alpha\in\mathbf{N}^k} r_{\alpha,\beta,j}(a)\frac{u^\alpha}{\alpha!}\right\} \cdot \frac{v^{\beta,j}}{\beta!}.$$

Then (6.10) implies that $\sum_\alpha r_{\alpha,\beta,j}(a)u^\alpha/\alpha!$ is the formal Taylor series at a of $r_{0,\beta,j}$. It follows from E. Borel's lemma [**22**, IV.3.5] that $R|(\Sigma_\mu - \Sigma_{\mu+1}) \in \mathscr{E}(\Sigma_\mu - \Sigma_{\mu+1})^p$. □

Proof of Lemma 6.7. We continue to use the notation in the proofs of Proposition 6.5 and Lemma 6.6. The assertion is local in Σ. Therefore, after perhaps shrinking U, we can assume there is a finite filtration of Σ by closed analytic subsets, $\Sigma = \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_{\nu+1} = Y$, such that, for all $\mu = 0,\ldots,\nu$:

(1) For each $i = 1,\ldots,s$, there exists $\psi^i \in \mathscr{O}(\Sigma_\mu)[[y]]^p$ such that ψ^i is a linear combination with coefficients in $\mathscr{O}(\Sigma_\mu)[y]$ of (the elements induced by) the columns ϕ^j of A and $\nu(\psi^i(a;\cdot)) = (\alpha_i, j_i)$, for all $a \in \Sigma_\mu - \Sigma_{\mu+1}$ (cf. Example 4.3(1) and Lemma 4.6).

Let $\mathscr{R}$ denote the subsheaf of $\mathscr{O}^q$ of (germs of) relations among the ϕ^j; i.e., q-tuples of germs of analytic functions $(\rho_1,\ldots,\rho_q)$ at points of U such that $\sum_{j=1}^q \rho_j \cdot \phi^j = 0$. Then $\mathscr{R}$ is coherent, so that $\mathfrak{N}(\hat{\mathscr{R}}_a)$ is Zariski semicontinuous on U. Therefore, we can also assume:

(2) $\mathfrak{N}(\hat{\mathscr{R}}_a)$ is constant, say $\mathfrak{N}(\hat{\mathscr{R}}_a) = \mathfrak{N}_\mu(\mathscr{R})$, on $\Sigma_\mu - \Sigma_{\mu+1}$.

(3) Let (β_i, k_i), $i = 1,\ldots,t$, denote the vertices of $\mathfrak{N}_\mu(\mathscr{R})$. Then, for each $i = 1,\ldots,t$, there exists σ^i in the submodule of $\mathscr{O}(\Sigma_\mu)[[y]]^q$ induced by $\mathscr{R}(U)$ such that $\nu(\sigma^i(a;\cdot)) = (\beta_i, k_i)$, for all $a \in \Sigma_\mu - \Sigma_{\mu+1}$.

By induction, it suffices to show that, for each $\mu = 0,\ldots,\nu$, $A \cdot \mathscr{E}(\Sigma_\mu; \Sigma_{\mu+1})^q = (A \cdot \mathscr{E}(\Sigma_\mu; \Sigma_{\mu+1})^q)^\wedge$.

Suppose $F \in (A \cdot \mathscr{E}(\Sigma_\mu; \Sigma_{\mu+1})^q)^\wedge$. As in the proof of Lemma 6.6, there exist $G_i \in \mathscr{E}(\Sigma_\mu; \Sigma_{\mu+1})$, $i = 1,\ldots,s$, such that $\hat{F}_a = \sum_{i=1}^s \hat{G}_{i,a} \cdot \theta_a^i$, for all $a \in \Sigma_\mu - \Sigma_{\mu+1}$. We will show that, for each $i = 1,\ldots,s$ and all $a \in \Sigma_\mu - \Sigma_{\mu+1}$,

$$\theta_a^i = \sum_{k=1}^q \eta_{k,a}^i \cdot \hat{\phi}_a^k, \tag{6.11}$$

where, for each i and k:

(i) There is an analytic function η_k^i defined in a neighborhood of $\Sigma_\mu - \Sigma_{\mu+1}$, whose power series expansion at each $a \in \Sigma_\mu - \Sigma_{\mu+1}$ is $\eta_{k,a}^i$.

(ii) The coefficients $\eta_{\alpha,k}^i(a)$ of $\eta_{k,a}^i(y) = \sum_\alpha \eta_{\alpha,k}^i(a) y^\alpha$ extend to Σ_μ as quotients of analytic functions whose denominators vanish nowhere in $\Sigma_\mu - \Sigma_{\mu+1}$.

Then, for each $k = 1, \ldots, q$, we put $H_{k,a} = \sum_i \eta^i_{k,a} \cdot \hat{G}_{i,a}$ if $a \in \Sigma_\mu - \Sigma_{\mu+1}$, and $H_{k,a} = 0$ if $a \in \Sigma_{\mu+1}$. By Lemma 6.3 (or directly from Łojasiewiczs's inequality), each $H_k = (H_{k,a})_{a \in \Sigma_\mu} \in \mathscr{E}(\Sigma_\mu; \Sigma_{\mu+1})$, and we have $F = A \cdot H$, where $H = (H_1, \ldots, H_q)$, as required.

To obtain (6.11): For each $i = 1, \ldots, s$, put $\psi^i_a(y) = \psi^i(a; y)$ and write $\psi^i_a(y) = \sum_{\alpha,j} \psi^i_{\alpha,j}(a) y^{\alpha,j}$. We first express each $\theta^i_a(y)$ in terms of the $\psi^l_a(y)$, using Theorem 3.1, then $\psi^l_a(y)$ in terms of the $\hat{\phi}^k_a(y) = \phi^k(a + y)$, by (1) above, to get $\theta^i_a(y) = \sum_{k=1}^q \xi^i_{k,a}(y) \cdot \hat{\phi}^k_a(y)$, where each $\xi^i_{k,a} \in \mathscr{O}_a$ and the coefficients $\xi^i_{\alpha,k}(a)$ of $\xi^i_{k,a}(y) = \sum_\alpha \xi^i_{\alpha,k}(a) y^\alpha$ are quotients of analytic functions by products of powers of the $\psi^l_{\alpha_l, j_l}(a)$.

Put $\xi^i_a = (\xi^i_{1,a}, \ldots, \xi^i_{q,a})$, $i = 1, \ldots, s$. By Theorem 3.1 and Remark 3.3, there exist unique $\eta^i_a \in \mathscr{O}^q_a$, $i = 1, \ldots, s$, such that each $\xi^i_a - \eta^i_a \in \mathscr{R}_a$ and $\operatorname{supp} \eta^i_a \cap \mathfrak{N}_\mu(\mathscr{R}) = \varnothing$. Write $\eta^i_a = (\eta^i_{1,a}, \ldots, \eta^i_{q,a})$ and $\eta^i_{k,a}(y) = \sum_\alpha \eta^i_{\alpha,k}(a) y^\alpha$, $k = 1, \ldots, q$. It follows from (3) above that each coefficient $\eta^i_{\alpha,k}(a)$ extends to Σ_μ as a quotient of an analytic function by a product of powers of the $\psi^l_{\alpha_l, j_l}(a)$ and the $\sigma^l_{\beta_l, k_l}(a)$ (where $\sigma^l(a; y) = \sum_{\beta,k} \sigma^l_{\beta,k}(a) y^{\beta,k}$). This gives (6.11), where the $\eta^i_{k,a}$ satisfy (ii).

To get (i): For b in some neighborhood of a in $\Sigma_\mu - \Sigma_{\mu+1}$, $\theta^i_b(y) = \theta^i_a(b - a + y)$ and $\hat{\phi}^k_b(y) = \hat{\phi}^k_a(b - a + y)$. From (6.11) it follows that

$$\sum_k \left(\eta^i_{k,b}(y) - \eta^i_{k,a}(b - a + y)\right) \cdot \hat{\phi}^k_b(y) = 0,$$

$i = 1, \ldots, s$. Since $\operatorname{supp}(\eta^i_b(y) - \eta^i_a(b - a + y)) \cap \mathfrak{N}_\mu(\mathscr{R}) = \varnothing$, then each $\eta^i_{k,b}(y) = \eta^i_{k,a}(b - a + y)$, by the uniqueness in Theorem 3.1, as required. □

7. Examples. The regular situation hypotheses of Theorem 6.4 seem about as good as can be expected for arbitrary closed sets X. But one might ask whether weaker hypotheses suffice when X is the closure of a domain with $\mathscr{C}^\infty$ boundary.

In each of the following examples, ϕ is an analytic function which generates the sheaf $\mathscr{I} = \mathscr{I}_\Sigma$ of germs of analytic functions vanishing of the zero set Σ of ϕ, and we consider the ideal in $\mathscr{E}(X)$ generated by ϕ. In the first five examples,

$$X = \left\{(t, x, y) \in \mathbf{R}^3 : x \geqslant e^{-1/t^2}\right\},$$

and it is easy to check that $\Sigma_1(\mathscr{I}) = \Sigma$, $\Sigma_2(\mathscr{I}) = \operatorname{Sing}\Sigma$ (the singular set of Σ) and $\Sigma_3(\mathscr{I}) = \varnothing$.

(1) Let $\phi(t, x, y) = x^3 + y^2$. Then Σ is not regularly situated with respect to X. The function e^{-1/t^2} belongs to $(\phi \cdot \mathscr{E}(X))^\wedge$, but not to $\phi \cdot \mathscr{E}(X)$ because, on the curve $\{y = 0,\ x = e^{-1/t^2}\}$, $e^{-1/t^2}/(x^3 + y^2) = x/x^3$ is not even continuous.

(2) Let $\phi(t, x, y) = x^3 - y^2$. Then Σ is regularly situated with respect to X, but $\operatorname{Sing}\Sigma$ is not. Nevertheless, $\phi \cdot \mathscr{E}(X) = (\phi \cdot \mathscr{E}(X))^\wedge$: Let $f(t, x, y) \in (\phi \cdot \mathscr{E}(X))^\wedge$. Consider the "second divided difference" $\nabla^2 f(t, x, y; y_1, y_2) \in \mathscr{E}(X \times \mathbf{R}^2)$ defined by the identity,

$$f(t, x, y) = f(t, x, y_1) + \frac{f(t, x, y_2) - f(t, x, y_1)}{y_2 - y_1} \cdot (y - y_1)$$
$$+ \nabla^2 f(t, x, y; y_1, y_2) \cdot (y - y_1)(y - y_2).$$

Substitute $y_1 = x^{3/2}$ and $y_2 = -x^{3/2}$ in this identity to get $f(t, x, y) = (y^2 - x^3) \cdot \nabla^2 f(t, x, y; x^{3/2}, -x^{3/2})$. Since $\nabla^2 f$ is symmetric with respect to (y_1, y_2), $\nabla^2 f = g(t, x, y; y_1 + y_2, y_1 y_2)$, where $g \in \mathscr{E}(X \times \mathbf{R}^2)$. Therefore, $f(t, x, y) = (y^2 - x^3) \cdot g(t, x, y; 0, -x^3)$.

(3) Let $\phi(t, x, y) = y^3 - x^2$. Then Σ is regularly situated with respect to X, but $\operatorname{Sing} \Sigma$ is not. Again, $\phi \cdot \mathscr{E}(X) = (\phi \cdot \mathscr{E}(X))^\wedge$: Let $\pi(t, x, v) = (t, x, v^2)$. Then $\phi \circ \pi = (v^3 - x)(v^3 + x)$ and $\pi^{-1}(X) = \{x \geqslant e^{-1/t^2}\}$. If $f(t, x, y) \in (\phi \cdot \mathscr{E}(X))^\wedge$, then $f(t, x, v^2) = (v^3 - x)(v^3 + x) \cdot h(t, x, v)$, where $h \in \mathscr{E}(\pi^{-1}(X))$, since each of $v^3 - x$ and $v^3 + x$ has gradient nowhere zero and zero set regularly situated with respect to $\pi^{-1}(X)$. But h is even in v, so that $h(t, x, v) = g(t, x, v^2)$, where $g \in \mathscr{E}(X)$; clearly, $f = \phi \cdot g$.

(4) Let $\phi(t, x, y) = y^3 - x^4$. Then Σ is regularly situated with respect to X, but $\operatorname{Sing} \Sigma$ is not. Again, $\phi \cdot \mathscr{E}(X) = (\phi \cdot \mathscr{E}(X))^\wedge$ since, after substituting $y = v^2$, $y^3 - x^4$ factors as $v^6 - x^4 = (v^3 - x^2)(v^3 + x^2)$ and the argument of (3) applies to each factor.

(5) Let $\phi(t, x, y) = y^3 - x^5$. Then Σ is regularly situated with respect to X, but $\operatorname{Sing} \Sigma$ is not. We do not know whether $\phi \cdot \mathscr{E}(X) = (\phi \cdot \mathscr{E}(X))^\wedge$! (See footnote 2.)

(6) Define $X \subset \mathbf{R}^4$ by $X = \{(s, t, x, y) : t \geqslant e^{-1/s^2}\}$. Let $\phi(s, t, x, y) = y^3 - tx^2$. Then $\Sigma_1(\mathscr{I}) = \Sigma$, $\Sigma_2(\mathscr{I}) = \operatorname{Sing} \Sigma = \{x = y = 0\}$ and $\Sigma_3(\mathscr{I}) = \{t = x = y = 0\}$; $\Sigma_1(\mathscr{I})$ and $\Sigma_2(\mathscr{I})$ are regularly situated with respect to X, but $\Sigma_3(\mathscr{I})$ is not. Nevertheless, $\phi \cdot \mathscr{E}(X) = (\phi \cdot \mathscr{E}(X))^\wedge$: the argument is similar to (3), taking π to be the blowing up of $\mathbf{R}^4$ with center $\{x = y = 0\}$.

8. Division in $\mathscr{A}^\infty(\Omega)$. We use the notation of the introduction. Let Ω be a domain with $\mathscr{C}^\infty$ boundary in $\mathbf{C}^n$, and let A be a $p \times q$ matrix with entries in $\mathscr{O}(\overline{\Omega})$.

LEMMA 8.1. $(A \cdot \mathscr{A}^\infty(\Omega)^q)^\wedge = \mathscr{A}^\infty(\Omega)^p \cap (A \cdot \mathscr{C}^\infty(\overline{\Omega})^q)^\wedge$.

PROOF. This is clear from the definitions in the introduction, since if B is a $p \times q$ matrix with entries in $\mathbf{C}[[z]] = \mathbf{C}[[z_1, \ldots, z_n]]$, then $B \cdot \mathbf{C}[[z]]^q = \mathbf{C}[[z]]^q \cap B \cdot \mathbf{C}[[z, \bar{z}]]^q$. □

Since $(A \cdot \mathscr{C}^\infty(\overline{\Omega})^q)^\wedge$ is a closed subspace of $\mathscr{C}^\infty(\overline{\Omega})^p$ containing $A \cdot \mathscr{C}^\infty(\overline{\Omega})^q$, $(A \cdot \mathscr{A}^\infty(\Omega)^q)^\wedge$ is a closed subspace of $\mathscr{A}^\infty(\Omega)^p$ which contains $A \cdot \mathscr{A}^\infty(\Omega)^q$.

Let U be an open neighborhood of $\overline{\Omega}$ in which the entries of A are holomorphic. Let $\mathscr{O} = \mathscr{O}_U$ and let $\mathscr{F} \subset \mathscr{O}^p$ denote the sheaf of $\mathscr{O}$-modules generated by the columns of A. By Proposition 5.2, the invariant diagram $\mathfrak{N}_a = \mathfrak{N}(\hat{\mathscr{F}}_a)$ is Zariski semicontinuous on U; i.e., there is a locally finite filtration of U by closed analytic subsets, $U = \Sigma_0(\mathscr{F}) \supset \Sigma_1(\mathscr{F}) \supset \cdots$, such that, for each $k = 0, 1, \ldots$,

(i) $\mathfrak{N}_a$ is constant, say $\mathfrak{N}_a = \mathfrak{N}_k(\mathscr{F})$, on $\Sigma_k(\mathscr{F}) - \Sigma_{k+1}(\mathscr{F})$;

(ii) $\mathfrak{N}_k(\mathscr{F}) < \mathfrak{N}_{k+1}(\mathscr{F})$.

THEOREM 8.2. *Suppose that Ω is a pseudoconvex domain with $\mathscr{C}^\infty$ boundary. If $\Sigma_k(\mathscr{F})$ and $\overline{\Omega}$ are regularly situated, for all $k \in \mathbf{N}$, then*

$$A \cdot \mathscr{A}^\infty(\Omega)^q = \left(A \cdot \mathscr{A}^\infty(\Omega)^q\right)^\wedge;$$

in particular, $A \cdot \mathscr{A}^\infty(\Omega)^q$ is a closed subspace of $\mathscr{A}^\infty(\Omega)^p$.

COROLLARY 8.3. *If Ω is a pseudoconvex domain with real analytic boundary, then* $A\cdot\mathscr{A}^\infty(\Omega)^q=(A\cdot\mathscr{A}^\infty(\Omega)^q)^\wedge$.

In the case that Ω is bounded, Theorem 8.2 is an immediate consequence of Theorem 6.4 and the following proposition.

PROPOSITION 8.4. *If Ω is a bounded pseudoconvex domain with $\mathscr{C}^\infty$ boundary in $\mathbf{C}^n$, then*

$$A\cdot\mathscr{A}^\infty(\Omega)^q=\mathscr{A}^\infty(\Omega)^p\cap A\cdot\mathscr{C}^\infty(\bar\Omega)^q.$$

We give a short (standard) proof of this below, using Kohn's theorem on global regularity in the $\bar\partial$-Neumann problem [**14**; **15**, §IV] (cf. [**20**]). Theorem 8.2, in general, is a consequence of Theorem 6.4 and the following proposition of Gay-Sebbar [**10**, Proposition 5.14].

PROPOSITION 8.5. *If Ω is a pseudoconvex domain with $\mathscr{C}^\infty$ boundary in $\mathbf{C}^n$, and $A\cdot\mathscr{C}^\infty(\bar\Omega)^q=(A\cdot\mathscr{C}^\infty(\bar\Omega)^q)^\wedge$, then $A\cdot\mathscr{A}^\infty(\Omega)^q=\mathscr{A}^\infty(\Omega)^p\cap A\cdot\mathscr{C}^\infty(\bar\Omega)^q$.*

To prove Proposition 8.4, we use two well-known lemmas:

LEMMA 8.6. *Let U and V be open subsets of $\mathbf{C}^n$ such that V is relatively compact and $\bar V\subset U$. Let $f^1,\dots,f^q\in\mathscr{O}(U)^p$. Then the subsheaf of $\mathscr{O}_V^q$ of relations among the $f^j\,|\,V$ is generated by finitely many global sections $g^i\in\mathscr{O}(U)^q$.*

PROOF. Let U^* denote the envelope of holomorphy of U [**12**, I.G]. Then each f^j extends to $f^j\in\mathscr{O}(U^*)^p$. Let $\mathscr{R}\subset\mathscr{O}_{U^*}^q$ denote the sheaf of relations among the f^j; $\mathscr{R}$ is coherent, by Oka's theorem [**12**, IV.C.1]. Since $\bar V$ is compact, it follows from Cartan's Theorem A [**12**, VIII.A.13] that there exist finitely many global sections g^i of $\mathscr{R}$ which generate $\mathscr{R}_a$, for all $a\in\bar V$. □

LEMMA 8.7 (CF. [**19**, LEMMA 5.7]). *Let $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$. Let U be an open subset of $\mathbf{K}^n$ and let $f^1,\dots,f^q\in\mathscr{O}(U)^p$. Suppose that the sheaf of relations among $f^1,\dots,f^q$ is generated by finitely many global sections $g^i\in\mathscr{O}(U)^q$. Let X be a closed subset of U, and let $\mu\colon\mathscr{E}(X)^q\to\mathscr{E}(X)^p$ denote the homomorphism $\mu(H)=\sum_{j=1}^q H_j\cdot f^j$, where $H=(H_1,\dots,H_q)\in\mathscr{E}(X)^q$. Then* $\operatorname{Ker}\mu$ *is generated by the g^i.*

PROOF. Let $H=(H_1,\dots,H_q)\in\operatorname{Ker}\mu$. Choose $h=(h_1,\dots,h_q)\in\mathscr{E}(U)^q$ such that $h\,|\,X=H$. Then $\Phi=\sum_j h_j\cdot f^j\in\mathscr{E}(U;X)^p$. By [**22**, V.2.4], we can factor Φ as $\Phi=\psi\cdot\Phi'$, where $\Phi'\in\mathscr{E}(U;X)^p$, $\psi\in\mathscr{E}(U;X)$ and $\psi>0$ on $U-X$. Clearly, Φ' formally belongs to the submodule of $\mathscr{E}(U)^p$ generated by $f^1,\dots,f^q$. Therefore, by Malgrange's theorem, $\Phi'=\sum_j h_j'\cdot f^j$, where each $h_j'\in\mathscr{E}(U)$. Now, $0=\Phi-\psi\cdot\Phi'=\sum_j(h_j-\psi\cdot h_j')\cdot f^j$. Again by Malgrange's theorem, $h-\psi\cdot h'$, where $h'=(h_1',\dots,h_q')$, is in the submodule of $\mathscr{E}(U)^q$ generated by the g^i. Since $\psi\in\mathscr{E}(U;X)$, then $H=h\,|\,X$ is in the submodule of $\mathscr{E}(X)^q$ generated by the g^i. □

PROOF OF PROPOSITION 8.4. Let $f^1,\dots,f^q$ denote the columns of A; the f^i are holomorphic in some neighborhood of $\bar\Omega$. By Lemma 8.6, there is an open neighborhood U of $\bar\Omega$ in $\mathbf{C}^n$, and an exact sequence of sheaves,

$$\mathscr{O}^{q_{n+1}}\xrightarrow{\mu_{n+1}}\mathscr{O}^{q_n}\to\cdots\to\mathscr{O}^{q_2}\xrightarrow{\mu_2}\mathscr{O}^{q_1}\xrightarrow{\mu_1}\mathscr{O}^p,$$

where $\mathcal{O} = \mathcal{O}_U$, $q_1 = q$, μ_1 is induced by the homomorphism $\mu_1(g_1, \ldots, g_q) = \sum_i g_i \cdot f^i$ on global sections, and, for each $j = 1, 2, \ldots$, $\operatorname{Ker}\mu_j$ is generated by global sections f_j^i, $i = 1, \ldots, q_{j+1}$, and μ_{j+1} is induced by the homomorphism $\mu_{j+1}(g_1, \ldots, g_{q_{j+1}}) = \sum_i g_i \cdot f_j^i$ on global sections.

For each $k = 0, \ldots, n$, let $\mathscr{E}^{0,k}(\overline{\Omega})$ denote the Fréchet space of $\mathscr{C}^\infty$ $(0, k)$-forms on $\overline{\Omega}$. Then $\mathscr{E}^{0,0}(\overline{\Omega}) = \mathscr{E}(\overline{\Omega}) = \mathscr{C}^\infty(\overline{\Omega})$. The exact sequence above induces a commutative diagram:

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathscr{A}^\infty(\Omega)^{q_{n+1}} & \overset{\mu_{n+1}}{\to} & \mathscr{A}^\infty(\Omega)^{q_n} & \overset{\mu_n}{\to} \cdots \to & \mathscr{A}^\infty(\Omega)^{q_1} & \overset{\mu_1}{\to} & \mathscr{A}^\infty(\Omega)^p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathscr{E}^{0,0}(\overline{\Omega})^{q_{n+1}} & \overset{\mu_{n+1}}{\to} & \mathscr{E}^{0,0}(\overline{\Omega})^{q_n} & \overset{\mu_n}{\to} \cdots \to & \mathscr{E}^{0,0}(\overline{\Omega})^{q_1} & \overset{\mu_1}{\to} & \mathscr{E}^{0,0}(\overline{\Omega})^p \\
\downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial \\
\mathscr{E}^{0,1}(\overline{\Omega})^{q_{n+1}} & \to & \mathscr{E}^{0,1}(\overline{\Omega})^{q_n} & \to \cdots \to & \mathscr{E}^{0,1}(\overline{\Omega})^{q_1} & \to & \mathscr{E}^{0,1}(\overline{\Omega})^p \\
\downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial \\
\mathscr{E}^{0,n}(\overline{\Omega})^{q_{n+1}} & \to & \mathscr{E}^{0,n}(\overline{\Omega})^{q_n} & \to \cdots \to & \mathscr{E}^{0,n}(\overline{\Omega})^{q_1} & \to & \mathscr{E}^{0,n}(\overline{\Omega})^p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

All rows except the first are exact, by Lemma 8.7. Each column is a direct sum of finitely many copies of the $\bar\partial$-sequence,

$$0 \to \mathscr{A}^\infty(\Omega) \to \mathscr{E}^{0,0}(\overline{\Omega}) \overset{\bar\partial}{\to} \mathscr{E}^{0,1}(\overline{\Omega}) \overset{\bar\partial}{\to} \cdots \overset{\bar\partial}{\to} \mathscr{E}^{0,n}(\overline{\Omega}) \to 0,$$

where $\bar\partial$ is determined by the formula

$$\bar\partial\left(g(z, \bar z)\, d\bar z_{i_1} \wedge \cdots \wedge d\bar z_{i_k}\right) = \sum_{l=1}^{n} \left(\frac{\partial g}{\partial \bar z_l}\right) \cdot d\bar z_l \wedge d\bar z_{i_1} \wedge \cdots \wedge d\bar z_{i_k}.$$

The $\bar\partial$-sequence, and hence each column, is exact by the theorem of Kohn [**14**; **15**, §IV]. A long diagram chase then shows that $\mathscr{A}^\infty(\Omega)^p \cap \mu_1(\mathscr{E}^{0,0}(\overline{\Omega})^{q_1}) = \mu_1(\mathscr{A}^\infty(\Omega)^{q_1})$, as required. □

9. Continuous linear division operators. Let Ω be a domain with $\mathscr{C}^\infty$ boundary in $\mathbf{C}^n$, and let A be a $p \times q$ matrix with entries in $\mathcal{O}(\overline{\Omega})$. Suppose that $A \cdot \mathscr{A}^\infty(\Omega)^q$ is a closed subspace of $\mathscr{A}^\infty(\Omega)^p$. Consider the exact sequence of Fréchet spaces,

$$0 \to \operatorname{Ker} A \to \mathscr{A}^\infty(\Omega)^q \to A \cdot \mathscr{A}^\infty(\Omega)^q \to 0.$$

Let s denote the space of *rapidly decreasing sequences* of real or complex numbers; i.e., sequences $x = (x_j)_{j \in \mathbf{Z}}$ such that, for every $k \in \mathbf{N}$, $\|x\|_k = \sup_j |j|^k |x_j| < \infty$. With the seminorms $\|\cdot\|_k$, s has the structure of a nuclear Fréchet space. The Fourier transform induces an isomorphism $\mathscr{C}^\infty(S^1) \cong s$. By a theorem of Vogt and Wagner [**24**, **25**], the sequence above admits a continuous linear splitting provided that $A \cdot \mathscr{A}^\infty(\Omega)^q$ is a closed subspace of s and $\operatorname{Ker} A$ is a quotient space of s.

Let M be a $\mathscr{C}^\infty$ manifold (with or without boundary). Then $\mathscr{C}^\infty(M)$ is a quotient of s. If M is compact, then $\mathscr{C}^\infty(M) \cong s$ [**4**, Proposition 6.3].

Assume that Ω is a bounded pseudoconvex domain with $\mathscr{C}^\infty$ boundary. There is an open neighborhood U of $\overline{\Omega}$ such that:

(1) The columns $f^1, \ldots, f^q$ of A extend to elements of $\mathscr{O}(U)^p$.

(2) (By Lemma 8.6.) Let $\mathscr{O} = \mathscr{O}_U$ and let $\mathscr{R} \subset \mathscr{O}^q$ be the sheaf of relations among $f^1, \ldots, f^q$. Then there exist $g^1, \ldots, g^r \in \mathscr{R}(U)$ which generate $\mathscr{R}_a$, for all $a \in U$.

Theorem 8.2 provides natural sufficient conditions under which $A \cdot \mathscr{A}^\infty(\Omega)^q$ is a closed subspace of $\mathscr{A}^\infty(\Omega)^p$, hence a closed subspace of $\mathscr{C}^\infty(\overline{\Omega})^p \cong s$.

It follows from (2) above that $\operatorname{Ker} A = ((g^1, \ldots, g^r) \cdot \mathscr{A}^\infty(\Omega))^\wedge$, where $(g^1, \ldots, g^r)$ denotes the submodule of $\mathscr{O}(U)^p$ generated by the g^i. By Proposition 8.4 and Lemma 8.7,

$$\operatorname{Ker} A = (g^1, \ldots, g^r) \cdot \mathscr{A}^\infty(\Omega);$$

hence $\operatorname{Ker} A$ is a quotient space of $\mathscr{A}^\infty(\Omega)^r$.

On the other hand, $\mathscr{A}^\infty(\Omega)$ (and therefore $\mathscr{A}^\infty(\Omega)^r$) is a quotient space of s under either of the following two conditions on Ω:

(a) $\partial\Omega$ is "weakly regular" in the sense of Catlin [**6**]. Then the $L^2(\Omega)$-orthogonal projection of $\mathscr{C}^\infty(\overline{\Omega})$ onto $\mathscr{A}^\infty(\Omega)$ is continuous [**6**], so that $\mathscr{A}^\infty(\Omega)$ is a quotient of $\mathscr{C}^\infty(\overline{\Omega}) \cong s$.

(b) $\partial\Omega$ is connected and Ω satisfies the conditions $Z(1)$, $Z(n-2)$ and $Z(n-1)$ of Folland and Kohn [**8**, p. 57]. Then, by [**8**, Theorem 5.3.2 and Corollary 5.4.13], the restriction to $\partial\Omega$ embeds $\mathscr{A}^\infty(\Omega)$ as a closed subspace of $\mathscr{C}^\infty(\partial\Omega)$, and the $L^2(\partial\Omega)$-orthogonal projection of $\mathscr{C}^\infty(\partial\Omega)$ onto $\mathscr{A}^\infty(\Omega)$ is continuous. Therefore, $\mathscr{A}^\infty(\Omega)$ is a quotient of $\mathscr{C}^\infty(\partial\Omega) \cong s$.

Every bounded strictly pseudoconvex domain Ω with $\mathscr{C}^\infty$ boundary is weakly regular.

Let U be an open neighborhood of $\overline{\Omega}$ in which the columns $f^1, \ldots, f^q$ of A are holomorphic, and let $\mathscr{O} = \mathscr{O}_U$. Let $\mathscr{F} \subset \mathscr{O}^p$ denote the sheaf of $\mathscr{O}$-modules generated by $f^1, \ldots, f^q$. We have proved:

COROLLARY 9.1. *Let Ω be a bounded pseudoconvex domain with $\mathscr{C}^\infty$ boundary, satisfying either condition* (a) *or* (b) *above. If each $\Sigma_k(\mathscr{F})$ is regularly situated with respect to $\overline{\Omega}$, then $A \cdot \mathscr{A}^\infty(\Omega)^q$ is a closed subspace of $\mathscr{A}^\infty(\Omega)^p$, and the canonical surjection $\mathscr{A}^\infty(\Omega)^q \to A \cdot \mathscr{A}^\infty(\Omega)^q$ admits a continuous linear splitting.*

10. $\mathscr{A}^\infty$ functions vanishing on an analytic set. Let Ω be a domain with $\mathscr{C}^\infty$ boundary in $\mathbf{C}^n$, and let X be a closed analytic subset of an open neighborhood of $\overline{\Omega}$. Set

$$\mathscr{A}^\infty(\Omega; X) = \{ g \in \mathscr{A}^\infty(\Omega) : g = 0 \text{ on } X \cap \Omega \}.$$

Let $\mathscr{I}_X$ denote the sheaf of germs of holomorphic functions which vanish on X. Suppose that $\mathscr{I}_X$ is generated by finitely many holomorphic functions $f^1, \ldots, f^q$ in a neighborhood U of $\overline{\Omega}$. Clearly, $\mathscr{A}^\infty(\Omega; X) \supset ((f^1, \ldots, f^q) \cdot \mathscr{A}^\infty(\Omega))^\wedge$. Theorem

8.2 gives natural sufficient conditions under which

$$\mathscr{A}^\infty(\Omega; X) = (f^1, \dots, f^q) \cdot \mathscr{A}^\infty(\Omega),$$

provided that $\mathscr{A}^\infty(\Omega; X) = ((f^1, \dots, f^q) \cdot \mathscr{A}^\infty(\Omega))^\wedge$.

REMARKS 10.1. (1) If $g \in \mathscr{A}^\infty(\Omega; X)$, then, for all $a \in \Omega$, $\hat{g}_a \in (\hat{f}_a^1, \dots, \hat{f}_a^q) \cdot \hat{\mathscr{O}}_a$; only for $a \in \partial\Omega$ may this condition fail.

(2) Let $\mathscr{C}^\infty(\bar{\Omega}; X \cap \bar{\Omega}) = \{g \in \mathscr{C}^\infty(\bar{\Omega})\colon g = 0 \text{ on } X \cap \bar{\Omega}\}$. Suppose $\overline{X \cap \Omega} = X \cap \bar{\Omega}$. Then $\mathscr{A}^\infty(\Omega; X) = \mathscr{A}^\infty(\Omega) \cap \mathscr{C}^\infty(\bar{\Omega}; X \cap \bar{\Omega})$. If $\mathscr{A}^\infty(\Omega; X) = (f) \cdot \mathscr{A}^\infty(\Omega)$, where $(f) = (f^1, \dots, f^q)$, then it is easy to see that $(f) \cdot \mathscr{A}^\infty(\Omega) = \mathscr{A}^\infty(\Omega) \cap (f, \bar{f}) \cdot \mathscr{C}^\infty(\bar{\Omega})$, where $\bar{f}^i$ denotes the conjugate of f^i. On the other hand, $\mathscr{C}^\infty(\bar{\Omega}; X \cap \bar{\Omega}) = (f, \bar{f}) \cdot \mathscr{C}^\infty(\bar{\Omega})$ only if $X \cap \bar{\Omega}$, with its underlying real structure, is coherent [**22**, VI.4.2]; compare this with (1) above and the following theorem.

THEOREM 10.2. *Assume that, for each* $k = 0, 1, \dots,$ *the closure of* $(\Sigma_k(\mathscr{I}_X) - \Sigma_{k+1}(\mathscr{I}_X)) \cap \Omega$ *contains* $(\Sigma_k(\mathscr{I}_X) - \Sigma_{k+1}(\mathscr{I}_X)) \cap \bar{\Omega}$. *Then*

$$\mathscr{A}^\infty(\Omega; X) = \left((f^1, \dots, f^q) \cdot \mathscr{A}^\infty(\Omega)\right)^\wedge.$$

PROOF. Suppose $g \in \mathscr{A}^\infty(\Omega; X)$. Then, for all $a \in \Omega$, $\hat{g}_a \in (\hat{f}_a^1, \dots, \hat{f}_a^q) \cdot \hat{\mathscr{O}}_a$. We have to show that, if $a \in \partial\Omega$, then $\hat{g}_a \in (\hat{f}_a^1, \dots, \hat{f}_a^q) \cdot \hat{\mathscr{O}}_a$.

If $a \in U$, put $\mathfrak{N}_a = \mathfrak{N}(\hat{\mathscr{I}}_{X,a})$ and $\mathfrak{G}_a = \mathfrak{G}(\hat{\mathscr{I}}_{X,a})$. Let $a_0 \in \partial\Omega$. Let $k \in \mathbf{N}$ such that $\Sigma_k(\mathscr{I}_X) = \{a \in U\colon \mathfrak{G}_a \geqslant \mathfrak{G}_{a_0}\}$. After a linear coordinate change, we can assume that $\mathfrak{N}_{a_0} = \mathfrak{G}_{a_0}$. Clearly, $\{a \in \Sigma_k(\mathscr{I}_X)\colon \mathfrak{N}_a = \mathfrak{N}_{a_0}\}$ is a Zariski open neighborhood of a_0 in $\Sigma_k(\mathscr{I}_X)$. Since $a_0 \in (\Sigma_k(\mathscr{I}_X) - \Sigma_{k+1}(\mathscr{I}_X)) \cup \bar{\Omega}$, it follows from the hypothesis that a_0 belongs to the closure of $\{a \in \Omega\colon \mathfrak{N}_a = \mathfrak{N}_{a_0}\}$.

Let α_i, $i = 1, \dots, t$, denote the vertices of $\mathfrak{N}_{a_0}$. Let $S = \{a \in U\colon \mathfrak{N}_a = \mathfrak{N}_{a_0}\}$ and, for each $a \in S$, let $g_a^i(z) \in \hat{\mathscr{O}}_a = \mathbf{C}[[z]]$, $i = 1, \dots, t$, denote the standard basis of $\hat{\mathscr{I}}_{X,a}$. Write $g_a^i(z) = \sum_{\alpha \in \mathbf{N}^n} g_\alpha^i(a) z^\alpha$. Then, by Corollary 3.4, each $g_a^i \in \mathscr{O}_a$ and the functions $g_\alpha^i(a)$ are analytic on S.

Suppose $a \in S \cap \bar{\Omega}$. By Theorem 3.1, there exist unique $q_{i,a}(z) \in \hat{\mathscr{O}}_a$, $i = 1, \dots, t$, and $r_a(z) \in \hat{\mathscr{O}}_a$ such that $\alpha_i + \operatorname{supp} q_{i,a} \subset \Delta_i$, $i = 1, \dots, t$, and $\operatorname{supp} r_a \subset \Delta$ (where the Δ_i, Δ are as in §3), and

$$\hat{g}_a(z) = \sum_{i=1}^t q_{i,a}(z) g_a^i(z) + r_a(z). \tag{10.3}$$

Write $r_a(z) = \sum_\alpha r_\alpha(a) z^\alpha$. Then each $r_\alpha(a)$ is continuous on S (cf. Remark 3.2). By (10.3), $\hat{g}_a \in \hat{\mathscr{I}}_{X,a} = (\hat{f}_a^1, \dots, \hat{f}_a^q) \cdot \hat{\mathscr{O}}_a$ if and only if each $r_\alpha(a) = 0$. Therefore, each $r_\alpha(a) = 0$ for $a \in S \cap \Omega$. Since $a_0 \in \overline{S \cap \Omega}$ and $a_0 \in S$, it follows that $\hat{g}_{a_0} \in (\hat{f}_{a_0}^1, \dots, \hat{f}_{a_0}^q) \cdot \hat{\mathscr{O}}_{a_0}$, as required. □

REMARK 10.4. The $\mathscr{A}^\infty$ extension problem. Let $\mathscr{A}^\infty(X \cap \Omega)$ denote the holomorphic functions on $X \cap \Omega$ which are restrictions of $\mathscr{C}^\infty$ functions on $\bar{\Omega}$. It is interesting to find natural sufficient conditions under which:

(1) the restriction homomorphism $\mathscr{A}^\infty(\Omega) \to \mathscr{A}^\infty(X \cap \Omega)$ is surjective, so there is an exact sequence

$$0 \to \mathscr{A}^\infty(\Omega; X) \to \mathscr{A}^\infty(\Omega) \to \mathscr{A}^\infty(X \cap \Omega) \to 0;$$

(2) $\mathscr{A}^\infty(X \cap \Omega)$ is a closed subspace of s.

For example, (1) and (2) hold if X is smooth and transverse to $\partial\Omega$ (cf. [**1**]). Suppose that Ω satisfies either condition (a) or (b), and that $\mathscr{A}^\infty(\Omega; X) = (f^1, \dots, f^q)\cdot\mathscr{A}^\infty(\Omega)$, as above. Then $\mathscr{A}^\infty(\Omega; X)$ is a quotient of s. If (1) and (2) hold, then the exact sequence in (1) splits, by the theorem of Vogt and Wagner.

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